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# LOGARITHMIC SOBOLEV INEQUALITY AND SEMI-LINEAR DIRICHLET PROBLEMS FOR INFINITELY DEGENERATE ELLIPTIC OPERATORS

*by*

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***Abstract.*** — Let $X = (X_1, \ldots, X_m)$ be an infinitely degenerate system of vector fields, we prove firstly the logarithmic Sobolev inequality for this system on the associated Sobolev function spaces. Then we study the Dirichlet problem for the semilinear problem of the sum of square of vector fields $X$.

***Résumé*** **(Inégalité de Sobolev logarithmique et problèmes de Dirichlet semi-linéaires pour des opérateurs elliptiques infiniment dégénérés)**

Soit $X = (X_1, \ldots, X_m)$ un système de champs de vecteurs infiniment dégénérés. On montre d'abord l'inégalité de Sobolev logarithmique pour ce système de champs de vecteurs sur les espaces de fonctions associés, puis on étudie le problème de Dirichlet semi-linéaire pour des opérateurs somme de carrés de champs de vecteurs $X$.

## 1. Introduction

In this work, we consider a system of vector fields $X = (X_1, \ldots, X_m)$ defined on an open domain $\widetilde{\Omega} \subset \mathbb{R}^d$. We suppose that this system satisfies the following logarithmic regularity estimate,

$$(1.1) \qquad \|(\log \Lambda)^s u\|_{L^2}^2 \leqslant C \left\{ \sum_{j=1}^{m} \|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}, \quad \forall u \in C_0^\infty(\widetilde{\Omega}),$$

where $\Lambda = (e + |D|^2)^{1/2} = \langle D \rangle$. We shall give some sufficient conditions for this estimates in the Appendix, see also [**5, 10, 12, 14, 15, 21**]. The typical example is the system in $\mathbb{R}^2$ such as $X_1 = \partial_{x_1}, X_2 = e^{-|x_1|^{-1/s}} \partial_{x_2}$ with $s > 0$. Remark that if $s > 1$, the estimate (1.1) implies the hypoellipticity of the infinitely degenerate elliptic operators of second order $\triangle_X = \sum_{j=1}^m X_j^* X_j$, where $X_j^*$ is the formal adjoint of $X_j$.

If $\Gamma$ is a smooth surface of $\widetilde{\Omega}$, we say that $\Gamma$ is non characteristic for the system of vector fields $X$, if for any point $x_0 \in \Gamma$, there exists at least one vector field of

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$X_1, \dots, X_m$ which is transversal to $\Gamma$ at $x_0$. Let now $\Gamma = \cup_{j\in J}\Gamma_j$ be the union of a family of smooth surface in $\widetilde{\Omega}$. We say that $\Gamma$ is non characteristic for $X$, if for any point $x_0 \in \Gamma$, there exists at least one vector field of $X_1, \dots, X_m$ which traverses $\Gamma_j$ at $x_0$ for all $j \in J_0 = \{k \in J; x_0 \in \Gamma_k\}$. For this second case, the typical example is $X_1 = \partial_{x_1}, X_2 = \exp(-(x_1^2 \sin^2(\pi/x_1))^{-1/2s})\partial_{x_2}$, we have $\Gamma_j = \{x_1 = 1/j\}$, $j \in \mathbb{Z} \setminus \{0\}, \Gamma_0 = \{x_1 = 0\}$, and $X_1$ is transverse to all $\Gamma_j, j \in \mathbb{Z}$.

Associated with the system of vector fields $X = (X_1, \dots, X_m)$, we define the following function spaces:

$$H^1_X(\widetilde{\Omega}) = \left\{u \in L^2(\widetilde{\Omega}); X_j u \in L^2(\widetilde{\Omega}), j = 1, \dots, m\right\}.$$

Take now $\Omega \subset\subset \widetilde{\Omega}$, we suppose that $\partial\Omega$ is $C^\infty$ and non characteristic for the system of vector fields $X$. We define $H^1_{X,0}(\Omega) = \{u \in H^1_X(\Omega); u|_{\partial\Omega} = 0\}$, we shall prove in the second section (see Lemma 2.1) that this is a Hilbert space.

Our first result is the following logarithmic Sobolev inequality.

***Theorem 1.1***. — *Suppose that the system of vector fields $X = (X_1, \dots, X_m)$ verifies the estimate (1.1) for some $s > 1/2$. Then there exists $C_0 > 0$ such that*

$$\int_\Omega |v|^2 \log^{2s-1}\left(\frac{|v|}{\|v\|_{L^2}}\right) \leqslant C_0\Big\{\sum_{j=1}^m \|X_j v\|^2_{L^2} + \|v\|^2_{L^2}\Big\}, \tag{1.2}$$

*for all $v \in H^1_{X,0}(\Omega)$.*

Comparing this inequality with that of finite degenerate case of Hörmander's system, for example, for the system $X_1 = \partial_{x_1}, X_2 = x_1^k\partial_{x_2}$ on $\mathbb{R}^2$, we have (see [**4, 7, 24**])

$$\|v\|_{L^p} \leqslant C\left(\|\partial_1 v\|^2_{L^2} + \|x_1^k\partial_2 v\|^2_{L^2} + \|v\|^2_{L^2}\right)^{1/2}$$

for all $v \in C_0^\infty(\Omega)$, with $p = 2 + 4/k$. Consequently, if $k$ go to infinity, we can only expect to gain the logarithmic estimates as (1.2). That means that we are not in the elliptic case of [**17**].

Similarly to the elliptic and subelliptic case (see [**3, 24**]), by using the Sobolev's inequality, we study the following semi-linear Dirichlet problems

$$\begin{aligned} &\triangle_X u = au\log|u| + bu, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{1.3}$$

where $a, b \in \mathbb{R}$. We have the following theorem.

***Theorem 1.2***. — *We suppose that the system of vector fields $X = (X_1, \dots, X_m)$ satisfies the following hypotheses:*

*H-1) $\partial\Omega$ is $C^\infty$ and non characteristic for the system of vector fields $X$ ;*

*H-2) the system of vector fields $X$ satisfies the finite type of Hörmander's condition on $\widetilde{\Omega}$ except an union of smooth surfaces $\Gamma$ which are non characteristic for $X$.*

*H-3) the system of vector fields $X$ verifies the estimate (1.1) for $s > 3/2$.*

*Suppose $a \neq 0$ in (1.3). Then the semi-linear Dirichlet problem (1.3) posses at least one non trivial weak solution $u \in H^1_{X,0}(\Omega) \cap L^\infty(\Omega)$. Moreover, if $a > 0$, we have $u \in C^\infty(\Omega \smallsetminus \Gamma) \cap C^0(\overline{\Omega} \smallsetminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \smallsetminus \Gamma$.*

As in the elliptic case, we do not know the uniqueness of solutions (see [**3**]). The regularity of this weak solution near to the infinitely degenerate point of $\Gamma$ is a more complicated problem, which will be studied in our future works.

The structure of the paper is as follows: The second section consists of the proof of Theorem 1.1. The third section is devoted to the proof for the existence of weak solution of Theorem 1.2, we introduce a variational problem and prove that the associated Euler-Lagrange equation is (1.3). In the fourth section we study the boundedness of weak solution of variational problems, which is a difficult step as in the classical case for the critical semilinear elliptic equations (see [**20**]). In the appendix we give some sufficient conditions for the logarithmic regularity estimates.

## 2. Logarithmic Sobolev inequality

We study now the function spaces $H^1_{X,0}(\Omega)$, see the similar results in [**22**].

***Lemma 2.1.*** — *Suppose that $\partial\Omega$ is $C^\infty$ and non characteristic for the system $X$, then $H^1_{X,0}(\Omega)$ is well-defined, and a Hilbert space. Moreover the extension of an element of $H^1_{X,0}(\Omega)$ by 0 belongs to $H^1_X(\widetilde{\Omega})$.*

*Proof.* — For the well-definedness, we need to prove the existence of trace for $v \in H^1_X(\Omega)$. We know that the trace problem is a local problem, so after the localization and straightened, we transfer the problem to the case: $v \in L^2(\mathbb{R}^d_+), \partial_{x_d} v \in L^2(\mathbb{R}^d_+)$ with support of $v$ is a subset of $\{|(x', x_d)| < c, x_d \geqslant 0\}$, of course we can take the smooth function approximate to $v$, then we have

$$v(x', x_d) - v(x', c) = \int_c^{x_d} \partial_{x_d} v(x', t)dt,$$

which prove that

$$\|v(\cdot, x_d)\|^2_{L^2} \leqslant c\|\partial_{x_d} v\|^2_{L^2}, \tag{2.1}$$

for all $0 \leqslant x_d \leqslant c$. This shows that the trace $v(x', 0) \in L^2(\mathbb{R}^{d-1})$.

We shall prove now $H^1_{X,0}(\Omega)$ is a closed subspace of $H^1_X(\Omega)$. Let $\{v_j\}$ be a Cauchy sequence of $H^1_{X,0}(\Omega)$. Since it is also a Cauchy sequence of $H^1_X(\Omega)$, there exists a limit $v_0 \in H^1_X(\Omega)$, and so it suffices to show that $v|_{\partial\Omega} = 0$. Applying (2.1) to $v_j - v_0$, we have

$$\|v_j(\cdot, 0) - v_0(\cdot, 0)\|^2_{L^2} \leqslant c\|\partial_{x_d}(v_j - v_0)\|^2_{L^2},$$

which implies $\|v_0(\cdot, 0)\|_{L^2} = 0$. We have proved that $H^1_{X,0}(\Omega)$ is a Hilbert space. The extension problem is the same as classic case. This is also a local problem, if we extend $v$ by 0 to $x_d < 0$ and denote that function by $\overline{v}$, then $v, \partial_{x_d} v \in L^2(\mathbb{R}^d_+), v|_{x_d=0} = 0$

implies that $\overline{v}, \partial_{x_d}\overline{v} \in L^2(\mathbb{R}^d)$, and the tangential derivation has nothing to change. So we have proved the Lemma.

Since $L\log L$ is not a normed space, we need the following Lemma, see also [**19**] for some detail of function space $L\log L$.

***Lemma 2.2.*** — *Let $\sigma_2 > 0$, $B > 0$ and let $\{v_j, j \in \mathbb{N}\}$ be a sequence in $L^2$ satisfying*

$$\int |v_j|^2 |\log |v_j||^{\sigma_2} \leqslant B.$$

*Then $\{|v_j|^2|\log|v_j||^{\sigma_1}\}$ is uniformly integrable for any $0 \leqslant \sigma_1 < \sigma_1$. Therefore there exists a convergent sub-sequence $\{v_{j_k}\}$ such that*

$$\lim_{k\to\infty} \int |v_{j_k}|^2 |\log |v_{j_k}||^{\sigma_1} = \int |v_0|^2 |\log |v_0||^{\sigma_1},$$

*and*

$$\int |v_0|^2 |\log |v_0||^{\sigma_2} \leqslant B.$$

*Proof.* — We prove that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \subset \Omega$, $\mu(E) < \delta$, then

$$\int_E |v_j|^2 |\log |v_j||^{\sigma_1} < \varepsilon, \quad \forall j.$$

But for any $\varepsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log^{\sigma_2-\sigma_1} t} < \varepsilon, \qquad \forall t \geqslant t_0.$$

Take now $\delta = \varepsilon(t_0^2 \log^{\sigma_1} t_0)^{-1}$, $\mu(E) < \delta$, and

$$A_j = E \cap \{|v_j| \leqslant t_0\}, \ B_j = E \cap \{|v_j| > t_0\}.$$

then

$$\int_{A_j} |v_j|^2 |\log |v_j||^{\sigma_1} \leqslant t_0^2 \log^{\sigma_1} t_0 \mu(A_j) < \varepsilon,$$

and

$$\int_{B_j} |v_j|^2 |\log |v_j||^{\sigma_1} \leqslant \varepsilon \int_{B_j} |v_j|^2 |\log |v_j||^{\sigma_2} \leqslant \varepsilon M$$

where $M = \sup_j \int_\Omega |v_j|^2 |\log |v_j||^{\sigma_2}$. The proof of the Lemma is complete.

*Proof of Theorem 1.1.* — We are following the idea of [**4**]. Take $v \in H^1_{X,0}(\Omega)$, we use the same notation for the extension by 0, As in the classical case, there exists a mollifier family $\{\rho_\varepsilon, \varepsilon > 0\}$ such that $\rho_\varepsilon * v \in C_0^\infty$, $\lim_{\varepsilon\to 0} \rho_\varepsilon * v = v$ in $L^2$ and $\|X(\rho_\varepsilon * v)\|_{L^2} \leqslant C\{\|Xv\|_{L^2} + \|v\|_{L^2}\}$, $\|(\log\Lambda)^s(\rho_\varepsilon * v)\|_{L^2} \leqslant C\{\|(\log\Lambda)^s v\|_{L^2} + \|v\|_{L^2}\}$ with $C$ independent on $\varepsilon$. By using (1.1) and Lemma 2.2, we need only to prove the following estimate:

$$\int_\Omega |v|^2 \log^{2s-1}\left(\frac{|v|}{\|v\|_{L^2}}\right) \leqslant C_0 \|(\log\Lambda)^s v\|^2_{L^2}, \tag{2.2}$$