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GROUP VELOCITY AT SMOOTH POINTS OF HYPERBOLIC CHARACTERISTIC VARIETIES

by

Jeffrey Rauch

To my friend Jean-Michel Bony with best wishes and appreciation for what he has taught me of mathematics other things.

Abstract. — At a smooth point of the characteristic variety defined by a homogeneous hyperbolic polynomial, the tangent plane determines the group velocity. In this note an algebraic algorithm is derived for computing this tangent plane at a given point. This is interesting only where the differential of the polynomial vanishes.

***Résumé* (Vitesse de groupe aux points lisses de variétés caractéristiques hyperboliques)**
En un point lisse d'une variété caractéristique définie par un polynôme homogène hyperbolique, le plan tangent détermine la vitesse de groupe. Dans cet article, on en déduit un algorithme algébrique de calcul de ce plan tangent en un point donné. Il n'est intéressant que là où la différentielle du polynôme s'annule.

Suppose that $P(D)$ is a homogeneous hyperbolic polynomial of degree $m \geqslant 1$ with time-like covector θ. Here $D = \partial/i\partial y$ with $y \in \mathbb{R}^n$. The symbol $P(\eta)$ is a homogeneous polynomial on $(\mathbb{R}^n)^*$. Hyperbolicity with respect to $\theta \in (\mathbb{R}^n)^*$ means that for any $\eta \in (\mathbb{R}^n)^*$ the equation

$$P(\eta + s\theta) = 0 \tag{1}$$

has only real roots s. In particular, $P(\theta) \neq 0$. Dividing P by $P(\theta)$ we may suppose that P has real coefficients ([**H**, Thm 8.7.3]).

The characteristic variety

$$\operatorname{Char} P := \{\eta \in \mathbb{R}^n \setminus 0 \,:\, P(\eta) = 0\}$$

is a conic real algebraic variety in $(\mathbb{R}^n)^*$. Since the equation (1) has m complex roots (counting multiplicity), and they all are real, it follows that every real line $\eta + s\theta$ intersects the variety in at least one point and no more than m points which shows

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that the variety has codimension 1 in $(\mathbb{R}^n)^*$. The fundamental stratification of real algebraic geometry (see [**BR**]) asserts that except for a set of codimension at least 2, the variety consists of smooth points, that is points where locally the variety is equal to the zero set of a real analytic function with nonvanishing gradient.

Definitions. — If $\underline{\eta} \neq 0$ is a point of the characteristic variety then $Q_{\underline{\eta}}(\eta)$ is the homogeneous polynomial of degree $k \geqslant 1$ which is the leading term in the expansion of $P(\underline{\eta}+\eta)$ about $\underline{\eta}$,

$$P(\underline{\eta}+\eta) = Q_{\underline{\eta}}(\eta) + \text{higher order terms in } \eta, \qquad Q_{\underline{\eta}} \neq 0.$$

At a smooth point $\underline{\eta}$, the annihilator of the tangent space $T_{\underline{\eta}}(\operatorname{Char} P)$ is a one dimensional linear subspace $L_{\underline{\eta}} \in \big(T_{\underline{\eta}}(\operatorname{Char} P)\big)^* = \mathbb{R}^n$. The lines in $\mathbb{R}^n$ parallel to $L_{\underline{\eta}}$ are those moving with the *group velocity* (see [**AR**]).

This velocity describes the propagation of wave packets, pulses, and singularities associated with the frequencies $(\mathbb{R} \setminus 0)\,\underline{\eta}$.

For variable coefficient operators, the above computations are performed in the tangent and cotangent spaces at a fixed point and P is the principal symbol at that point. They are pertinent for example for symmetric hyperbolic systems and points of the characteristic variety which are microlocally of constant multiplicity.

If $\underline{\eta} \in \operatorname{Char} P$ is a smooth point of multiplicity one, that is $P(\underline{\eta}) = 0$ and $dP(\underline{\eta}) \neq 0$, then $dP(\underline{\eta})$ is a basis for $L_{\underline{\eta}}$ and one has a simple way of recovering the velocity from the symbol.

In an analogous way, at a smooth point one can write the variety locally as $q = 0$ with $dq \neq 0$, then $dq(\underline{\eta})$ is a basis for $L_{\underline{\eta}}$. However, there is no algebraic algorithm to find a function q starting from the defining function P when the roots have multiplicity greater than one. The following two results provide a straightforward algorithm to compute the group velocity for hyperbolic operators. In a subsequent article, [**MR**], it is shown, by an independent calculation, that there are algebraic formulas yielding the entire germ of $\operatorname{Char} P$ at $\underline{\eta}$.

Theorem. — *If $\underline{\eta}$ is a smooth point of the characteristic variety and $Q_{\underline{\eta}}$ is as above, then there is a real linear form $\ell(\eta)$ so that the tangent plane at $\underline{\eta}$ to the characteristic variety of P is equal to $\{\ell(\eta-\underline{\eta}) = 0\}$, and, $Q_{\underline{\eta}}(\eta) = \operatorname{sign}(Q(\theta))\,\ell(\eta)^{\deg Q}$.*

Corollary. — *If $\underline{\eta}$ is a smooth point of the characteristic variety and $Q_{\underline{\eta}}$ and $L_{\underline{\eta}}$ are as above, then for all η which are not in the characteristic variety of $Q_{\underline{\eta}}$ (e.g. $\eta = \theta$), $dQ_{\underline{\eta}}(\eta)$ is a basis for $L_{\underline{\eta}}$.*

These proofs rely on the fundamental theorems concerning Local Hyperbolicity (see [**G**]). That theory is closely related to the ideas of microhyperbolicity introduced by Bony and Shapira in [**BS**] (see [**H**, § 8.7]).

Examples. — Consider $P(\eta_1, \eta_2) = (\eta_1^2 - \eta_2^2)^2$, the square of the wave operator with time-like $\theta = (1,0)$. At the smooth point $\underline{\eta} = (1,1)$, the conormal is easy to compute directly by considering the reduced equation $\eta_1^2 - \eta_2^2 = 0$. But, illustrating the above results, compute

$$P(\underline{\eta}+\eta) = \Big((1+\eta_1)^2 - (1+\eta_2)^2\Big)^2 = \Big(\eta_1^2 + 2\eta_1 - \eta_2^2 - 2\eta_2\Big)^2 = (2\eta_1 - 2\eta_2)^2 + \text{ h.o.t.}$$

Thus $\ell = 2\eta_1 - 2\eta_2$, $Q_{\underline{\eta}} = \ell^2$, $L_{\underline{\eta}}$ is generated by $(2,-2)$, and the group velocity is equal to -1.

The above example, the examples from [**H**] and [**C**], and the examples from mathematical physics that I know, all have the property that for any smooth point $\underline{\eta}$ there is an explicit hyperbolic factor of the symbol vanishing at $\underline{\eta}$ and with nonvanishing gradient. For those examples an appeal to the above algorithm can be avoided.

The proof of the Theorem begins with the fact from [**G**] that $Q_{\underline{\eta}}(\eta)$ is hyperbolic with time-like covector θ. Then for every real η the equation

$$Q_{\underline{\eta}}(\eta + s\theta) = 0 \tag{2}$$

has only real roots s.

Lemma 1. — *For every real η the equation (2) has exactly one root s.*

Proof. — With $k :=$ the degree of $Q_{\underline{\eta}}$, one has as $\varepsilon \to 0$,

$$\varepsilon^{-k} P(\underline{\eta} + \varepsilon(\eta + s\theta)) = Q_{\underline{\eta}}(\eta + s\theta) + O(\varepsilon). \tag{3}$$

If (2) had two roots s_1 and s_2, then Rouché's theorem would imply that the characteristic variety of P would have points near $\underline{\eta} + \varepsilon(\eta + s_j\theta)$ as $\varepsilon \to 0$ violating the smooth variety hypothesis. □

The next Lemma is then applied to $R = Q_{\underline{\eta}}$.

Lemma 2. — *If $R(\eta)$ is a homogeneous real polynomial hyperbolic with respect to the time-like covector θ and for all real η the equation $R(\eta + s\theta) = 0$ has exactly one real root s, then there is a real linear form $\ell(\eta)$ such that*

$$R(\eta) = \operatorname{sign}(R(\theta))\, \ell(\eta)^{\deg R}.$$

Proof. — Introduce coordinates $(\tau, \xi_1, \dots, \xi_{n-1})$ in $(\mathbb{R}^n)^*$ so that $\theta = (1, 0, \dots, 0)$. Then

$$R(\tau, \xi) = R(1, 0, \dots, 0)\big(\tau^k + a_1(\xi)\tau^{k-1} + \cdots + a_{k-1}(\xi)\tau + a_k(\xi)\big)$$

with $a_j(\xi)$ homogeneous of degree j and $k = \deg R \geqslant 1$.

By hypothesis, for each real ξ the equation $R(\tau, \xi) = 0$ has a unique root $\tau = r(\xi)$. Therefore

$$R(\tau, \xi) = R(1, 0, \dots, 0)\big(\tau - r(\xi)\big)^k.$$

Equating coefficients of τ^{k-1} shows that

$$-k\, r(\xi) = a_1(\xi),$$

so $r(\xi)$ is a homogeneous polynomial of degree 1. The Lemma follows with $\ell(\tau,\xi) = c(\tau - r(\xi))$ provided that c is chosen as

$$c := |R(1,0,\dots,0)|^{1/k} \quad \text{so } c^k = \operatorname{sign}(R(1,0,\dots,0))\, R(1,0,\dots,0). \qquad \square$$

Proof of Theorem. — Combining the above lemmas implies that

$$Q_{\underline{\eta}}(\eta) = \operatorname{sign}(Q(\theta))\, \ell(\eta)^k.$$

It remains to show that the tangent plane to the characteristic variety of P is given by the equation $\ell(\eta - \underline{\eta}) = 0$.

Use the local coordinates (τ,ξ) from the proof of Lemma 2. Since $\theta = (1,0,\dots,0)$ is noncharacteristic for P, the variety of P is given by the roots τ of $P(\tau,\xi) = 0$ with ξ ranging over $\mathbb{R}^n \setminus 0$.

The points near $\underline{\eta} = (\underline{\tau},\underline{\xi})$ are then given by the roots τ of

$$P(\underline{\tau} + \varepsilon\tau, \underline{\xi} + \varepsilon\xi) = 0, \tag{4}$$

with $|\xi| \leqslant 1$. Equation (2) takes the form

$$\varepsilon^{-k}\, P(\underline{\tau} + \varepsilon\tau, \underline{\xi} + \varepsilon\xi) = Q_{\underline{\eta}}(\tau,\xi) + O(\varepsilon). \tag{5}$$

Since $Q_{\underline{\eta}} = \ell^k$, the equation $Q_{\underline{\eta}}(\tau,\xi) = 0$ is equivalent to the equation $\ell(\tau,\xi) = 0$. Since $\ell(\theta)^k = Q_{\underline{\eta}}(\theta) \neq 0$, it follows that the solutions of $\ell(\tau,\xi) = 0$ are given by $\tau = \underline{x}\cdot\xi$ for an appropriate $\underline{x}$.

Rouché's Theorem applied to (5) shows that for $|\xi| < 1$ the roots of (4) are given by

$$\tau = \underline{x}\cdot\xi + O(\varepsilon).$$

The corresponding points $\eta = (\underline{\tau} + \varepsilon\tau, \underline{\xi} + \varepsilon\xi)$ of the characteristic variety of P differ from $\underline{\eta}$ by $O(\varepsilon)$ and satisfy

$$\ell(\eta - \underline{\eta}) = O(\varepsilon^2).$$

This shows that the equation of the tangent plane is $\ell(\eta - \underline{\eta}) = 0$. $\square$

Proof of Corollary. — Since $Q_{\underline{\eta}} = \pm\ell^k$ one has

$$dQ_{\underline{\eta}}(\eta) = \pm\, k\, \ell(\eta)^{k-1}\, d\ell(\eta).$$

Since ℓ is a linear form on $(\mathbb{R}^n)^*$, $d\ell(\eta)$ is a vector which does not depend on η. The Theorem implies that $d\ell$ is a basis for $L_{\underline{\eta}}$. Therefore, $dQ_{\underline{\eta}}(\eta)$ is a basis whenever it is nonvanishing. This holds exactly for η which satisfy $\ell(\eta) \neq 0$ which are exactly those η which are not in the characteristic variety of $Q_{\underline{\eta}}$. $\square$