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GEOMETRY OF MULTI-DIMENSIONAL DISPERSING BILLIARDS

by

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Abstract. — Geometric properties of multi-dimensional dispersing billiards are studied in this paper. On the one hand, non-smooth behaviour in the singularity submanifolds of the system is discovered (this discovery applies to the more general class of semi-dispersing billiards as well). On the other hand, a self-contained geometric description for unstable manifolds is given, together with the proof of important regularity properties. All these issues are highly relevant to studying the ergodic and statistical behaviour of the dynamics.

1. Introduction

Let $\mathbb{Q}$ be an open connected domain in $\mathbb{R}^d$ or on the d-dimensional torus $\mathbb{T}^d$. Assume that the boundary $\partial\mathbb{Q}$ consists of a finite number of C^k smooth ($k \geqslant 3$) compact hypersurfaces (possibly, with boundary). Now let a pointwise particle move freely (along a geodesic line with constant velocity) in $\mathbb{Q}$ and reflect elastically at the boundary $\partial\mathbb{Q}$ (by the classical rule "the angle of incidence is equal to the angle of reflection"). This is what is commonly refered to as a billiard dynamical system.

Billiards make an important class in the modern theory of dynamical systems. Many classical and quantum models in physics belong to this class, most notably, the Lorentz gas [**Si**] and hard ball gases studied as early as the XIX century by L. Boltzmann [**Bo**].

The periodic Lorentz process is obtained by fixing a finite number of disjoint convex bodies $B_1, \dots, B_s \subset \mathbb{T}^d$ with smooth boundary and putting the moving particle in

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the exterior domain $\mathbb{Q} = \mathbb{T}^d \setminus (\cup B_i)$. This system models the motion of an electron among a periodic array of molecules in a metal, as it was introduced by H. Lorentz in 1905.

Mathematical studies of billiards have begun long ago. Ya. Sinai in his seminal paper of 1970 [**Si**] described the first large class of billiards with truly chaotic behavior — with nonzero Lyapunov exponents, positive entropy, enjoying ergodicity, mixing, and (as was later discovered by G. Gallavotti and D. Ornstein [**GO**]) the Bernoulli property. Sinai billiards are defined in two dimensions ($d = 2$), i.e. for $\mathbb{Q} \subset \mathbb{R}^2$ or $\mathbb{Q} \subset \mathbb{T}^2$, and the boundary of $\mathbb{Q}$ must be concave (i.e., convex inward $\mathbb{Q}$), similarly to the Lorentz process (where the bodies B_i are convex). Due to the geometric concavity, the boundary $\partial\mathbb{Q}$ scatters or disperses bundles of geodesic lines falling upon it, see Fig. 1. For this reason, Sinai billiards are said to be *dispersing*.

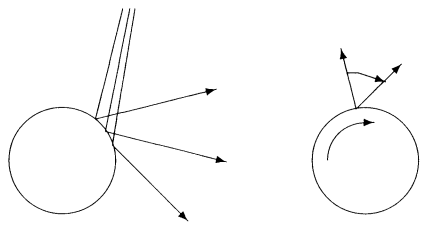

FIGURE 1. Scattering effect

Lorentz processes in two dimension have been studied very thoroughly since 1970. Many fine ergodic and statistical properties have been established by various researchers, including P. Bleher, L. Bunimovich, N. Chernov, J. Conze, C. Dettmann, G. Gallavotti, A. Krámli, J. Lebowitz, D. Ornstein, K. Schmidt, N. Simányi, Ya. Sinai, D. Szász, and others (see the references). The latest major result for this model (the exponential decay of correlations) was obtained by L.-S. Young [**Y1**]. The success in these studies had significant impact on modern statistical mechanics. The methods and ideas originally developed for the planar Lorentz process were applied to many other classes of physical models — see recent reviews by Cohen, Gallavotti, Ruelle and Young [**GC, Ru, Y2**].

On the other hand, the progress in the study of the multidimensional Lorentz process (where $d > 2$) has been much slower and somewhat controversial. Relatively few papers were published covering specifically the case $d > 2$, especially in contrast to the big number of works on the 2-D case. Furthermore, the arguments in the published articles were usually rather sketchy, as in Chernov's paper [**Ch1**]. It was commonly assumed that the geometric properties of the multidimensional Lorentz process were essentially similar to those of the 2-D system, and so the basic methods of study should be extended from 2-D to any dimension at little cost. Thus, the authors rarely elaborated on details.

Recent discoveries proved that spatial dispersing billiards are *very much* different from planar ones. Bunimovich and Reháček studies of *astigmatism* [**BR**], in the somewhat different context of focusing billiards, emphasized the known fact that the billiard trajectories may focus very rapidly in one plane and very slowly in the orthogonal planes. Astigmatism is unique to 3-D (and higher dimensional) billiards, it cannot occur on a plane. It plays an improtant role in higher dimensional focusing billiards as investigated in [**BR**].

In this paper we consider multidimensional *dispersing* billiards. We show that multi-dimensionality has great effect on the dynamics in the dispersing case as well — the system requires much more elaborated study than the 2D process. What is worse (cf. section 3), the singularity manifolds in the phase space of a spatial Lorentz process have pathologies — points exist where the sectional curvature is unbounded (blows up). Actually, singularity manifolds are in these pathologies — which form two-codimensional submanifolds of them — not even differentiable. Indeed, as it will be shown in section 3, the unit normal vector to the singularity manifold has different directional limits at the pathological points — the geometry is pretty much like the classical Whitney umbrella $x^2z = y^2$ in $\mathbb{R}^3$. This phenomenon is again unique to billiards in dimension $d \geqslant 3$. All these facts call for a revision of some earlier arguments and results on the multidimensional Lorentz process. This is much the more important since the studies of physically relevant multiparticle systems will require the same methods as those used for the high-dimensional Lorentz process.

Throughout the paper we conduct a systematic study of the geometry of the Lorentz process in any dimension $d > 2$, aiming at the future investigation of its ergodic and statistical properties (in particular, the decay of correlations). First we describe our recent discovery — pathological behavior of singularity manifolds — and show exactly where it occurs (in order to "localize the pathology"). Then we develop tools for the study of basic geometric properties of the dynamics — operator techniques in the Poincaré section of the phase space. By applying these geometric tools we provide rigorous proofs of important properties for unstable manifolds: we show absolute continuity, distorsion bounds, curvature bounds and alignment. All these facts are absolutely important for the studies of ergodic and statistical properties of the Lorentz gas, but strangely enough, their proofs (in the case of dimension $d > 2$) have never been published before. Lastly, we show how our results can be used in the study of the decay of correlations, which will be done in a separate paper.

2. Preliminaries

There are two ways of considering billiard dynamics, the motion of a point particle in a connected, compact domain $\mathbb{Q} \subset \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geqslant 2$ with a piecewise C^3-smooth boundary. The phase space of the *flow* can be identified with the unit tangent bundle over $\mathbb{Q}$ — the configuration space is $\mathbb{Q}$ while the phase space is $\mathcal{M} := \mathbb{Q} \times \mathbb{S}^{d-1}$

($\mathbb{S}^{d-1}$ is the surface of the unit d-ball). In other words, every phase point x is of the form (q, v) where $q \in \mathbb{Q}$ and $v \in \mathbb{S}^{d-1}$. We denote the flow by $S^t : -\infty < t < \infty$.

On the other hand there is a naturally defined cross-section for this flow. The phase space of the Poincaré section *map* (or simply, of the billiard map) is $M := \partial\mathbb{Q} \times \mathbb{S}^{d-1}_+$, where $+$ means that we only take into account the hemisphere of the outgoing velocities (for a more precise definition of the phase space, see subsection 4.1). For any $x \in \mathcal{M}$ we set $t^+(x) := \inf\{t > 0 \mid S^t x \in M\}$, and $T^+ x := S^{t^+(x)} x$ (of course, $T^+ : \mathcal{M} \to M$). Then the Poincaré section map $T : M \to M$ is defined as follows: $Tx := T^+ x$ for $x \in M$.

We require the following properties from the system to be studied:

– Our billiard is *dispersing* (a Sinai-billiard): each $\partial\mathbb{Q}_i$ is strictly convex (had we required convexity only, our billiard would be *semi-dispersing*).

– The scatterers B_i are disjoint. This ensures the C^3-smoothness of the boundary $\partial\mathbb{Q}$, i.e. that there are *no corner points*.

– The condition that *the horizon is finite* says exactly that $t^+(x) < \infty$ for any $x \in M$.

Finally, some more notation. Let $n(q)$ be the unit normal vector of the boundary component $\partial\mathbb{Q}_i$ at $q \in \partial\mathbb{Q}_i$ directed inwards $\mathbb{Q}$. Then the invariant Liouville-measure of the discretized map is

$$d\mu(q, v) := \text{const.}\, \langle n(q), v\rangle\, dq\, dv \tag{2.1}$$

where dq is the induced Riemannian measure on $\partial\mathbb{Q}$ whereas dv is the Lebesgue-measure on $\mathbb{S}^{d-1}_+$.

Throughout the paper, unless otherwise emphasized, we are considering this discretized dynamics.

2.1. Fronts. — In billiard theory, several basic constructions and concepts are based on the notion of a local orthogonal manifold, which - for simplicity - we will call front. A front $\mathcal{W}$ is defined in the whole phase space rather than in the Poincaré section. Take a smooth 1-codim submanifold E of the whole configuration space, and add the unit normal vector $v(r)$ of this submanifold at every point r as a velocity, continuously. Consequently, at every point the velocity points to the same side of the submanifold E. Then

$$\mathcal{W} = \{(r, v(r)) \mid r \in E\} \subset \mathcal{M},$$

where $v : E \to \mathbb{S}^{d-1}$ is continuous (smooth) and $v \perp E$ at every point of E. The derivative of this function v, called B plays a crucial role: $dv = Bdr$ for tangent vectors (dr, dv) *of the front*. B acts on the tangent plane $\mathcal{T}_r E$ of E, and takes its values from the tangent plane $\mathcal{J} = \mathcal{T}_{v(r)}\mathbb{S}^{d-1}$ of the velocity sphere. These are both naturally embedded in the configuration space $\mathbb{Q}$, and can be identified through this embedding. So we just write $B : \mathcal{J} \to \mathcal{J}$. B is nothing else than the curvature operator of the submanifold E. Yet we will prefer to call it second fundamental