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ON THE SCALING STRUCTURE FOR PERIOD DOUBLING

by

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Abstract. — We describe an order on the set of scaling ratios of the generic universal smooth period doubling Cantor set and prove that this set of ratios forms itself a Cantor set, a Conjecture formulated by Coullet and Tresser in 1977. This result establishes explicitly the geometrical complexity of the universal period doubling Cantor set. We also show a convergence result for the two period doubling renormalization operators, acting on the codimension one space of period doubling maps. In particular they form an iterated function system whose limit set contains a Cantor set.

1. Definitions and Statement of the Results

A *unimodal map with critical exponent* $\alpha > 1$ is an interval map that can be written in the form $f = \psi \circ q_t \circ \phi$, where ψ and ϕ are orientation preserving C^3 diffeomorphisms of $[0, 1]$, and $q_t : [0, 1] \to [0, 1]$ with $t \in (0, \frac{1}{2}]$ is the *standard folding map (with critical exponent* $\alpha > 1$) defined by

$$q_t(x) = 1 - \frac{|x - t|^\alpha}{|1 - t|^\alpha},$$

that "folds" the interval at its unique critical point t, $q_t(t) = 1$ and $q_t^2(t) = 0$.

The space of orientation preserving diffeomorphisms of the interval $[0, 1]$ with fixed smoothness is denoted by $\mathrm{Diff}^k([0, 1])$. The space of unimodal maps with fixed critical

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exponent $\alpha > 1$ and fixed smoothness can be represented by

$$\mathcal{U} = \mathrm{Diff}^k([0,1]) \times (0, \tfrac{1}{2}] \times \mathrm{Diff}^k([0,1]).$$

It carries what we call *C^k-distances d_k*, $k \geqslant 3$, which combines the two C^k distances on each of the two diffeomorphisms ψ and ϕ with the distance between the parameters t of the folding parts. Notice that in general, the critical point of f is $c_f = \phi^{-1}(t) \neq t$. Let p_f be the unique fixed point of $f \in \mathcal{U}$. A map on the interval is *renormalizable* if it exchanges some number N_1 of subintervals. The return map on one of these subintervals can again be renormalizable, exchanging this time N_2 intervals. If the process continues forever, one says the map is *infinitely renormalizable*. For precise definitions and an account of the theory, see for instance [**dMvS**]. Except otherwise specified when we say renormalizable, we mean renormalizable in the sense of period doubling, *i.e.*, the map exchanges two intervals. We will only consider infinitely renormalizable maps with $N_1 = N_2 = \cdots = 2$.

Fix a critical exponent $\alpha > 1$. We consider the set W of maps $f : [0,1] \to [0,1]$ with $f(c_f) = 1$ and $f(1) = 0$ which are infinitely renormalizable. The critical point defines two invariant intervals

$$U_f = [f^2(c_f), f^4(c_f)] \quad \text{and} \quad V_f = [f^3(c_f), f(c_f)].$$

To these two intervals correspond two *renormalization operators* $R_0 : W \to W$ and $R_1 : W \to W$ defined by:

$$R_0 f = [f^2|V_f], \quad \text{and} \quad R_1 f = [f^2|U_f],$$

where $[\cdot]$ means *affine rescaling to obtain a unimodal map on* $[0,1]$ *that sends its critical point to 1 and 1 to 0.*

Observe, both operators preserve W and R_1 is the critical point period doubling renormalization operator which has been most studied in the literature (see in particular [**La**], [**Ly**], [**Mc**], [**dMvS**], [**S2**], and references therein for the case when α is an even integer, and [**E1**], [**E2**] and [**Ma2**] for arbitrary $\alpha > 1$).

Let T_n be the set of all words of length n over the alphabet $\{0,1\}$. We denote by T the set of all infinite words of the form $w1^\infty$ over the alphabet $\{0,1\}$, and by $\overline{T}$ the set of all infinite words over the alphabet $\{0,1\}$, equipped with the usual metric. Notice that each T_n naturally embeds into T. For any word $\tau \in \overline{T}$, we will write $\tau_{\{n\}} \in T_n$ for the initial segment of length n of τ. We are going to consider the iterated function system generated by R_0 and R_1. To this end, we define:

$$R_{\tau_{\{n\}}} = R_{\tau(1)} \circ \cdots \circ R_{\tau(n)} : W \longrightarrow W,$$

and we will prove the following convergence result for this iterated function system.

Theorem 1.1. — *For any fixed point f_0 of R_0, there is a Hölder-continuous map $h : \overline{T} \to W$ such that for any $\tau \in \overline{T}$*

$$\lim_{n\to\infty} R_{\tau_{\{n\}}} f_0 = h(\tau).$$

Moreover, the convergence of the sequence $\{R_{\tau_{\{n\}}} f_0\}$ *is exponential in the* C^2*-metric.*
A similar statement holds for any fixed point f_1 *of* R_1.

Remark 1.2. — For any $\alpha > 1$, the existence of a fixed point f_1 of R_1 is proven in [**E1, E2**] and [**Ma2**]. We will show (see Lemma 2.4) that the existence of a fixed point f_1 for R_1 is equivalent to the existence of a fixed point f_0 for R_0. The uniqueness of f_1 in the case when α is an even integer was proven in [**S2**]. In the sequel we will fix f_0 and f_1 to be fixed points of respectively R_0 and R_1.

Remark 1.3. — The set $h(\overline{T})$ of limits $\lim_{n\to\infty} R_{\tau_{\{n\}}} f_0$ is denoted by $A \subset W$. Here the notation A represents the fact that we believe, but do not prove, that the set A is indeed the attractor of the iterated function system generated by R_0 and R_1, and in particular does not depend on the initial point, chosen here to be f_0.

The second Main result, Theorem 1.10, describes the structure of the set A in the case when $\alpha = 2$. It relies on convexity properties of f_0 and $R_1(f_0)$.

Convexity Conditions 1.4. — *We assume that:*

C1 $\quad f_0|[(f_0)^3(c_{f_0}), 1)$ *is strictly convex,*

C2 $\quad R_1(f_0)|[(R_1(f_0))^3(c_{R_1(f_0)}), 1]$ *is strictly convex.*

Remark 1.5. — In section 4 we will show that **C1** actually holds true in the case when successive R_1 renormalizations of a convex function converge to f_1: this is known to be the case when α is an even integer. Furthermore, as we will explain, one can check that both **C1** and **C2** hold true in the most important case of generic (quadratic) critical points, $\alpha = 2$.

Recall that a Cantor set is a perfect and totally disconnected compact metric space.

Proposition 1.6. — *If the Convexity Conditions C1 and C2 hold true, then the limit set* A *of orbits of* f_0 *under the interated function system defined by* R_0 *and* R_1 *is a Cantor set.*

For completeness and to fix notations and definitions, we include some basic discussion of the scaling function, whose origin is rather diffuse: first conjectures about a form of it appeared in [**CT**], the name and a form of it come from [**F**], while what was arguably the first theorem about it was in a never circulated work by Feigenbaum and Sullivan cited in [**S1**]. The literature on scaling functions is extensive and discusses scaling functions beyond the context of dynamics. In particular, in [**KSV**] a relation with the thermodynamic formalism appeared.

Let Λ be the invariant Cantor set of f_0. In the sequel we will remind the dynamical construction of covers of Λ by finitely many intervals. These covers, called cycles, form a refining nest of covers of this Cantor set. The scaling function contains the infinitesimal geometrical information on how these covers refine. It will be shown that the Cantor set Λ is, from a geometrical point of view, very different from the well known middle third Cantor set, in which each refinement is done everywhere in the same manner.

Although, the Cantor set Λ is the invariant set of a non expanding map, it is also the invariant Cantor set of an expanding interval map, the so-called *presentation function* [**R**], [**S1**], a great remark that Rand attributes to Misiurewicz. As we next recall, this directly follows from f_0 being a renormalization fixed point that is expanding to the right of p_{f_0}.

Let $U = U_{f_0}$ and $V = V_{f_0} = [1 - v, 1]$. The affine (scaling) map $s : [0,1] \to [0,1]$ defined by $s : x \mapsto v \cdot (x - 1) + 1$ is a homeomorphism from Λ to $\Lambda \cap V$. This is a direct consequence of the fact that s conjugates $f_0 = R_0(f_0) = s^{-1} \circ f_0^2 \circ s$ to f_0^2. Also the restriction,

$$f_0|V : \Lambda \cap V \longrightarrow \Lambda \cap U,$$

is a homeomorphism so that the map $g : [0,1] \to U$ defined by $g = (f_0|V) \circ s$ is a homeomorphism from Λ to $\Lambda \cap U$. Let $F : [0,1] \to [0,1]$ be the multivalued function defined by the two branches

$$F_0 = s : [0,1] \longrightarrow [0,1] \quad \text{and} \quad F_1 = g : [0,1] \longrightarrow [0,1].$$

The branch $F_0 = s$ is affine, contracting, and orientation preserving while the branch $F_1 = g$ is orientation reversing. Furthermore, the absolute value of the derivative of F_1 strictly increases as a consequence of the Convexity Condition C1, so that F_1 is also contracting (as p_{f_0} is an expanding fixed point). It follows that the invariant set of the iterated function system $F = \{F_0, F_1\}$ is Λ, the invariant Cantor set of f_0.

The cover $\{U, V\}$ of Λ is called *the cycle of the first generation.* The two intervals of this cycle are permuted by the map f_0. The Cantor set Λ is the intersection of a decreasing sequence of covers we call respectively the *cycles of generation* n: the cycle of generation n is the cover of Λ consisting of 2^n intervals which are permuted by f_0. The intervals that form the n^{th} cycle can be described as follows.

The construction of the cycles is made by using the iterated function system generated by F_0 and F_1. We will use a notation for the words describing sequences of compositions of these maps that will be different from the one we used in the definition of the iterated function system generated by R_0 and R_1. Namely, we write Σ_n for the set of words $w = w(1)w(2)\dots w(n)$ of length $|w| = n$ over the alphabet $\{0,1\}$, and Σ for the set of infinite sequences over the alphabet $\{0,1\}$ with the usual metric. Let

$$I_w = F_{w(n)} \circ \cdots \circ F_{w(1)}([0,1]).$$

The n^{th} cycle consists of the intervals I_w with w a word of length n.

Lemma 1.7. — *The way f_0 permutes these intervals is described by addition* mod 2^n *on the words indexing the intervals. In particular, if c is the critical point of f_0 then*