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ROBUSTLY TRANSITIVE SETS AND HETERODIMENSIONAL CYCLES

by

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Abstract. — It is known that all non-hyperbolic robustly transitive sets Λ_φ have a dominated splitting and, generically, contain periodic points of different indices. We show that, for a $\mathcal{C}^1$-dense open subset of diffeomorphisms φ, the indices of periodic points in a robust transitive set Λ_φ form an interval in $\mathbb{N}$. We also prove that the homoclinic classes of two periodic points in Λ_φ are robustly equal. Finally, we describe what sort of homoclinic tangencies may appear in Λ_φ by studying its dominated splittings.

1. Introduction

When a diffeomorphism ϕ is *hyperbolic*, i.e., it verifies the Axiom A, the Spectral Decomposition Theorem of Smale says that its limit set (set of non-wandering points) is the union of finitely many *basic pieces* satisfying *nice* properties, each piece is invariant, compact, *transitive* (i.e., it contains an orbit which is a dense subset), pairwise disjoint and isolated (each piece is the maximal invariant set in a neighborhood of itself). Moreover, by construction, a basic piece is the *homoclinic class* of a hyperbolic periodic point, i.e., the closure of the transverse intersections of its invariant manifolds.

Even if the dynamics is non-hyperbolic, the homoclinic classes of hyperbolic periodic points seem to be the natural elementary pieces of the dynamics, satisfying many of the properties of the basic sets of the Smale's theorem: invariance, compactness, transitivity and density of hyperbolic periodic points. Recent results in [**BD**$_2$], [**Ar**] and [**CMP**] show that, for $\mathcal{C}^1$-generic diffeomorphisms (i.e., those belonging to

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a residual subset of $\mathrm{Diff}^1(M)$) two homoclinic classes are either disjoint or equal and they are maximal transitive sets (i.e., every transitive set intersecting a homoclinic class is contained in it). Notice that, in general, the homoclinic classes fail to be hyperbolic, isolated and pairwise disjoint.

In [**BDP**] it is shown that, for $\mathcal{C}^1$-generic diffeomorphisms, a homoclinic class is either contained in the closure of an infinite set of sinks or sources, or satisfies some weak form of hyperbolicity (partial hyperbolicity or, at least, existence of a *dominated splitting*). The first situation (called the *Newhouse phenomenon*) can be locally generic, in the residual sense: there exist open sets in $\mathrm{Diff}^r(M)$ where the diffeomorphisms with infinitely many sinks or sources are (locally) residual for the $\mathcal{C}^r$-topology. The case $r \geqslant 2$ for surface diffeomorphisms can be found in [**N**], see [**PV**] for the case $r \geqslant 2$ in higher dimensions, and [$\mathbf{BD}_1$] for $r = 1$ in dimensions greater than or equal to 3. Certainly, the Newhouse phenomenon exhibits very wild behavior and it is conjectured that (in some sense) diffeomorphisms satisfying this phenomenon are very rare (for instance, for generic parametrized families of diffeomorphisms, the Lebesgue measure of the parameters corresponding to diffeomorphisms satisfying the Newhouse phenomenon is zero), see [**Pa**].

We focus here on the opposite behavior. More precisely, we restrict our attentions to the so-called *robustly transitive sets* introduced in [**DPU**] as a non-hyperbolic generalization of the basic sets of the *Spectral Decomposition of Smale.* A *robustly transitive set* Λ of a diffeomorphism ϕ is a transitive set which is locally maximal in some neighbourhood U of it and such that, for every C^1-perturbation ψ of the diffeomorphism ϕ, the maximal invariant set of ψ in U is transitive. From the results in [$\mathbf{M}_2$], [**DPU**] and [**BDP**] every robustly transitive set Λ admits a *dominated splitting,* say $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$, and by [$\mathbf{BD}_2$], $\mathcal{C}^1$-generically, it is a *homoclinic class.* An invariant set may admit more than one dominated splitting, since one can always sum up some bundles of the original dominated splitting, obtaining a new dominated splitting with less bundles, or, conversely, split some bundle of the splitting in a dominated way. So it is natural to consider the *finest dominated splitting* of the set Λ (i.e., the one that does not admit any dominated sub-splitting).

In this paper we study the interrelation between the dominated splittings (especially the finest one) of a robustly transitive set Λ and its dynamics, answering questions about the *indices* (dimension of the stable manifold) of the periodic points of Λ, the possible bifurcations (saddle-node and homoclinic tangencies) occurring in this set as well as its dynamical structure.

Let us recall some definitions, necessary for what follows.

In what follows, M denotes a compact, closed Riemannian manifold and $\mathrm{Diff}^1(M)$ the space of $\mathcal{C}^1$-diffeomorphisms of M endowed with the usual topology.

Let Λ be a compact invariant set of a diffeomorphism ϕ. A ϕ_*-invariant splitting $T_\Lambda M = E \oplus F$ over Λ is said to be *dominated* if the fibers of E and F have constant dimension and there exists $k \in \mathbb{N}$ such that, for every $x \in \Lambda$, one has

$$||\phi_*^k|_{E(x)}|| \cdot ||\phi_*^{-k}|_{F(\phi^k(x))}|| < \frac{1}{2},$$

that is, the vectors in F are uniformly more expanded than the vectors in E by the action of ϕ_*^k. If it occurs we say that F *dominates* E and write $E \prec F$.

An invariant bundle E over Λ is *uniformly contracting* if there exists k such that, for every $x \in \Lambda$, one has:

$$||\phi_*^k|_{E(x)}|| < \frac{1}{2}.$$

An invariant bundle E over Λ is *uniformly expanding* if it is uniformly contracting for ϕ_*^{-1}.

Let $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ be a ϕ_*-invariant splitting over Λ such that the fibers of the bundles E_i have constant dimension. Denote by $E_i^j = \bigoplus_i^j E_k$ the direct sum of $E_i, \dots, E_j$. Note that $E_1^{k-1} \oplus E_k^m$ is a splitting of $T_\Lambda M$ for all $k \in \{2, \dots, m\}$. We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_m$ is the *finest dominated splitting* of Λ if $E_1^{k-1} \oplus E_k^m$ is a dominated splitting for each $k \in \{2, \dots, m\}$ and every E_k is *indecomposable* (i.e., it does not admit any nontrivial dominated splitting). See [**BDP**] for the existence and uniqueness of the finest dominated splitting.

Consider a set $V \subset M$ and a diffeomorphism $\varphi \colon M \to M$. We denote by $\Lambda_\varphi(V)$ the *maximal invariant set* of φ in V, i.e., $\Lambda_\varphi(V) = \bigcap_{i \in \mathbb{Z}} \varphi^i(V)$. Given an open set $U \subset M$ the set $\Lambda_\varphi(U)$ is *robustly transitive* if $\Lambda_\psi(U)$ is equal to $\Lambda_\psi(\overline{U})$ and is *transitive* for all ψ in a $\mathcal{C}^1$-neighbourhood of φ. We say that a ψ-invariant closed set K is *transitive* if there exists some $x \in K$ having a positive orbit which is dense in K.

If a robustly transitive set $\Lambda_\phi(U)$ is not (uniformly) hyperbolic then, by a $\mathcal{C}^1$-small perturbation of ϕ, one can create non-hyperbolic periodic points, and thus hyperbolic periodic points with different indices in $\Lambda_\phi(U)$ (see [$\mathbf{M}_2$]). Our first two results describe the possible indices of the periodic points of $\Lambda_\phi(U)$, in terms of the finest dominated splitting of $\Lambda_\phi(U)$:

Theorem A. — *Let U be an open set of M and $\mathcal{M}(U)$ a $\mathcal{C}^1$-open subset of* $\mathrm{Diff}^1(M)$ *such that $\Lambda_\varphi(U)$ is robustly transitive for every $\varphi \in \mathcal{M}(U)$. Then there is a dense open subset $\mathcal{N}(U)$ of $\mathcal{M}(U)$ such that, for every $\varphi \in \mathcal{N}(U)$, the set of indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$ is an interval of integers (i.e., if P and Q are hyperbolic periodic points of indices p and q, $p \geqslant q$, of $\Lambda_\varphi(U)$, $\varphi \in \mathcal{N}(U)$, and $j \in [q,p]$, then $\Lambda_\varphi(U)$ has a hyperbolic periodic point of index j).*

In the next result, we use the arguments in [$\mathbf{M}_2$] to relate the uniform contraction or expansion of the extremal bundles of the finest dominated splitting of a robustly transitive set with the indices of the periodic points of this set.

Theorem B. — *Consider an open subset U of a compact manifold M and an integer $q \in \mathbb{N}^*$. Let $\mathcal{U}$ be a $\mathcal{C}^1$-open subset of* $\mathrm{Diff}^1(M)$ *such that for every $\phi \in \mathcal{U}$ the maximal invariant set $\Lambda_\phi(\overline{U})$ satisfies the following properties:*

(1) *the set* $\Lambda_\phi(\overline{U})$ *is contained in* U *and admits a dominated splitting* $E_\phi \oplus F_\phi$, $E_\phi \prec F_\phi$, *with* $\dim E_\phi(x) = q$,

(2) *the set* $\Lambda_\phi(\overline{U})$ *has no periodic points of index* $k < q$.

Then the bundle E_ϕ *is uniformly contracting for every* $\phi \in \mathcal{U}$.

We can summarize the two results above, in order to get a characterization of the set of indices of the periodic points of the set $\Lambda_\phi(\overline{U})$, as follows.

Let $U \subset M$ be open and φ a diffeomorphism such that $\Lambda_\varphi(U)$ is robustly transitive with a finest dominated splitting of the form $T_{\Lambda_\varphi(U)}M = E_1 \oplus \cdots \oplus E_{k(\varphi)}$, $E_i \prec E_{i+1}$. Denote by E^s the sum of all uniformly contracting bundles of this splitting and let E_α be the first non-uniformly contracting bundle, i.e., $E^s = E_1 \oplus \cdots \oplus E_{\alpha-1}$. In the same way, denote by E^u the sum of all uniformly expanding bundles of the splitting and let E_β be the last non-uniformly expanding bundle, i.e., $E^u = E_{\beta+1} \oplus \cdots \oplus E_{k(\varphi)}$. Let $\mathcal{U}$ be a $\mathcal{C}^1$-neighborhood of φ such that, for every $\psi \in \mathcal{U}$, the set $\Lambda_\psi(U)$ has the same properties as $\Lambda_\varphi(U)$ (i.e., robustly transitive and the number $k(\psi)$ of bundles of the finest dominated splitting is equal to $k(\varphi)$) and the dimensions of bundles $E^s(\psi)$, $E_\alpha(\psi)$, $E_\beta(\psi)$ and $E^u(\psi)$, defined in the obvious way, are constant in $\mathcal{U}$ and equal to corresponding bundles for ϕ.

Corollary C. — *With the notation above, there exist a* $\mathcal{C}^1$*-open and dense subset* $\mathcal{V}$ *of* $\mathcal{U}$ *and locally constant functions* $i, j\colon \mathcal{V} \to \mathbb{N}^*$ *such that*

$$i(\psi) \in [\dim(E^s), \dim(E^s) + \dim(E_\alpha)] \cap \mathbb{N}^*,$$
$$j(\psi) \in [\dim(E^u), \dim(E^u) + \dim(E_\beta)] \cap \mathbb{N}^*,$$

and, for every $\psi \in \mathcal{V}$, *the set of indices of the hyperbolic periodic points of* $\Lambda_\psi(\overline{U})$ *is the interval* $[i(\psi), \dim(M) - j(\psi)] \cap \mathbb{N}^*$.

The first known examples of non-hyperbolic robustly transitive sets had a one-dimensional central direction, see [$\mathbf{M}_1$] and [**Sh**]. As a consequence, these examples do not present *homoclinic tangencies* (non-transverse homoclinic intersections between the invariant manifolds of some periodic point). Observe that if a periodic point has a homoclinic tangency then, after a perturbation of the diffeomorphism, one create a Hopf bifurcation (a periodic point whose derivative has a pair of conjugate nonreal eigenvalues of modulus one), see [**YA**] and [**R**], hence points whose central direction has dimension at least two. Currently examples of robustly transitive sets having a central direction of dimension two or more are known, see [$\mathbf{BD}_1$], [**B**] and [**BV**]. Moreover, in some cases these sets exhibit homoclinic tangencies, see [**B**] and [**BV**]. Our next result explains what sort of dominated splitting of a robustly transitive set prevents homoclinic bifurcations.

We say that a robustly transitive set $\Lambda_\varphi(U)$ is $\mathcal{C}^1$*-far from homoclinic tangencies* if there are no *homoclinic tangencies* in $\Lambda_\psi(U)$, for all ψ in a $\mathcal{C}^1$-neighbourhood of φ.