Astérisque

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Astérisque, tome 286 (2003), p. 1-24
<http://www.numdam.org/item?id=AST_2003__286__1_0>

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286, 2003, p. 1–24

ON THE MATHEMATICAL CONTRIBUTIONS OF JACOB PALIS

by

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Abstract. — A Conference on Dynamical Systems celebrating the 60th birthday of Jacob Palis was held at IMPA (Instituto de Matemática Pura e Aplicada) in Rio de Janeiro from July 19-28, 2000. This article is a revised and expanded version of a lecture I gave at the Conference. Many additions, including the list of references and the entire sections below on *Homoclinic Bifurcations*, *Cantor Sets and Fractal Invariants*, *Non-Hyperbolic Systems*, and *A Unifying View of Dynamics*, were made later by Marcelo Viana. It was decided to preserve the flavor of the lecture by keeping the narrative in the first person. I am grateful to Marcelo for his contributions to this paper. In my opinion, they greatly improved the presentation of the mathematical scope and influence of Jacob Palis.

Introduction

Let me begin just by saying that Jacob has made many, many contributions to Mathematics. I will not talk about all of them because, in fact, in one hour it's impossible to discuss in any detail all of them. I pick some of what I consider to be the main contributions, and there will be relatively little that is new for experts, but I hope you will be reminded of many experiences during the last thirty or some years of the development of Dynamical Systems.

First, to my mind his primary mathematical contributions fit into three categories:

– global stability related to the concepts of structural stability and Ω-stability;

– bifurcation theory, which is how systems depending on parameters change, how their structure changes.

– formulation of some general ideas and conjectures, that motivated several very interesting recent results in this field.

I will talk about these aspects of his work a little bit later. Besides these types of subjects there are many other ancillary results, many interesting kinds of things.

2000 ***Mathematics Subject Classification.*** — 37Dxx.
Key words and phrases. — Hyperbolic, Morse-Smale, structural stability, Axiom A, bifurcation theory, homoclinic tangency, tubular families, Ω-stability, stability conjecture.

But, together with the mathematical contributions that he has been making, one has to appreciate and understand the overview and direction of research that Jacob is responsible for. At the present time he is at

– 35 graduate students, and some 30 grand-students, originating from 10 different countries mainly in Latin America, as you can see in his academic tree (attached to this paper).

Some of these students have become main figures in the whole theory of Dynamical Systems, in fact in the world of Mathematics. You know who they are as well as I do, I don't need to mention names. It's a testimony to his vision, his generosity, and the freedom of ideas that he's encouraged, that he is such an inspiration to so many people.

In addition, I think it's really fair to say that in our time Jacob Palis has been one of the main figures responsible for the development of Mathematics and Science, primarily in Latin America(1) and, in fact, in many other places, through his

– organization of meetings, symposia, workshops, and the support of sciences and Mathematics in developing countries, most notably, that I'm familiar with, in Trieste.

He has facilitated the contacts between scientists who have had great difficulty in traveling to the west for political or other reasons. They were able to establish contacts with western mathematicians in the settings of meetings, workshops, and schools where one can get to meet many people. I myself met a number of people from mainland China in Trieste, at a time when it was extremely difficult for them to travel to Western Europe. Jacob has been one of the primary organizers and supporters of such occasions.

Moreover, he has been responsible, in great measure, for

– the tremendous growth of IMPA, this wonderful institute, as a researcher and, more recently, also as the Director.

I think it's fair to say that IMPA has become the principal center for Mathematics in Latin America and, certainly, one of the world centers for Dynamical Systems. In no small measure is this due to his efforts and, again, his vision.

I want to go now toward some of the mathematical developments Jacob has accompanied in his many years of activity.

Structural Stability

Let me go back to 1960. Let M be a compact connected smooth manifold without boundary, and let us consider the space $\mathcal{D}^r(M)$ of C^r diffeomorphisms on M, and the

(1)The impact of Jacob Palis's work throughout Latin America was the subject of another lecture at the Conference, by Alberto Verjovsky.

space $\mathcal{X}^r(M)$ of C^r vector fields on M, as well as certain distinguished well-known subsets of these

$\mathcal{D}^r_{ss}(M)$ = set of C^1 structurally stable diffeomorphisms on M,
$\mathcal{X}^r_{ss}(M)$ = set of C^1 structurally stable vector fields on M.

This notion of *structural stability* means that under any small C^1 perturbation, the entire orbit structure persists after a global continuous coordinate change. As far as I know, it was first presented by Andronov and Pontrjagin in 1937. They introduced these systems, that they called rough systems, or coarse systems, and the primary part of the paper [**2**] was to characterize them among vector fields in the two dimensional disk which were nowhere tangent to the boundary. And what they described in that paper was that a vector field X is structurally stable if and only if

(a) X has only finitely many critical elements (singular points and periodic orbits), all hyperbolic,

(b) and there are no saddle connections.

The next principal result connected to structural stability we will mention was due to Maurício Peixoto in a paper [**53**] that was published in 1959. There, he studied various general properties of structurally stable systems and proved that the Andronov-Pontrjagin systems formed an open and dense subset of the set of all vector fields on the two dimensional disk which were nowhere tangent to the boundary. Later, in [**54**], in a somewhat surprising way, he proved the following theorem: on a compact oriented surface M^2,

– the structurally stable vector fields $\mathcal{X}^r_{ss}(M^2)$ form a dense open set in the space $\mathcal{X}^r(M^2)$ and

– they are completely characterized by the Andronov-Pontrjagin conditions (a) and (b), and the additional condition that the α- and ω-limit sets of every point x are critical elements.

As far as I know, originally this paper was thought to prove that the result is true for all surfaces (not necessarily orientable), but that's still not known, except in the case of genus two, where Carlos Gutierrez [**18**] proved the general result, and in the C^1 topology, where it is a consequence of Pugh's closing-lemma [**56**].

This led to two main questions at the time:

– Is $\mathcal{X}^r_{ss}(M)$ non-empty, that is, do structurally stable systems exist on any manifold?

– Is $\mathcal{X}^r_{ss}(M)$ always dense in the space $\mathcal{X}^r(M)$ of all vector fields?

Also the analogous questions for C^r diffeomorphisms on compact manifolds.

Well, to some people's disappointment, the second question, the density, has a negative answer. That was discovered by Smale around 1964 or 65. He found out that on any manifold in dimension bigger than or equal to 4 there were open sets of vector fields which were not structurally stable. That dimension was then made optimal by Bob Williams in the end of the 60's [**68**]: he found more detailed versions of Smale's theorem, and a counter-example in dimension 3.

Around the same time, in the 60's, in the Soviet Union, Anosov studied other kinds of structurally stable systems. The systems that he called C-diffeomorphisms [**3**], where the entire space had a splitting into two continuous distributions invariant by the derivative, one of which was exponentially expanded and the other exponentially contracted under iterates. These systems, now well known, were coined the name Anosov diffeomorphisms by Smale in his 1967 paper [**65**] in the Bulletin of the AMS. What Anosov was able to to prove for these systems was that

– they formed an open subset of the set of all C^1 diffeomorphisms on a manifold
– and they were structurally stable systems.

The methods were related (I don't know, in fact, in which order) to his celebrated result that geodesic flows on manifolds with negative curvature were structurally stable and had the flow version of these Anosov conditions.

At this time, in the mid 60's, what was then the status of this kind of mathematics? We had high dimensional examples of structurally stable systems. They exhibited very complicated recurrence, and they were only known in special manifolds. In fact, for the Anosov systems the existence of the invariant bundles of course brings with it topological obstructions. So, for example on surfaces, Anosov diffeomorphisms only exist on the torus. And in higher dimensions, also only on very special manifolds. In fact, for a while it was felt that the only manifolds that admitted Anosov diffeomorphisms were the tori, of any dimension. Smale found examples using other kinds of Lie groups, non-Abelian Lie groups, but still they were very special in the kinds of manifolds that can exhibit them.

What about simple recurrence, that is, systems that don't have complicated recurrent orbits? Motivated by gradient systems, which Smale sort of used for going back and forward between dynamical systems and topology, a special class of dynamical systems, which we now call Morse-Smale systems, was defined. In the diffeomorphism case, these are systems where the non-wandering set consists of a finite number of hyperbolic periodic orbits, and if you have two such orbits their stable and unstable manifolds are transverse. Analogous definitions were given for vector fields, where the non-wandering set consists of finitely many critical points and periodic orbits all hyperbolic, and with the transversality conditions.

Smale was able to prove that there was a residual set of gradient systems (a residual set of functions) on any compact manifold that were Morse-Smale, and their time-one maps were Morse-Smale diffeomorphisms. The easy part of this is to realize that a Morse function has only hyperbolic critical points as its non-wandering set. But it's not so obvious to get the transversality condition: that is a consequence of a general approximation theorem, the Kupka-Smale theorem, which was done in those days. And Smale conjectured that,