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COUPLED HOPF-BIFURCATIONS: PERSISTENT EXAMPLES OF n-QUASIPERIODICITY DETERMINED BY FAMILIES OF 3-JETS

by

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Abstract. — In this note examples are presented of vector fields depending on parameters and determined by the 3-jet, which display persistent occurrence of n-quasiperiodicity. In the parameter space this occurrence has relatively large measure. A leading example consists of weakly coupled Hopf bifurcations. This example, however, is extended to full generality in the space of all 3-jets.

1. Introduction

In the theory of coupled reaction diffusion equations the following is of interest, see Poláčik *et al.* [**12, 19**]. The problem is whether persistent examples exist of (parameter dependent) dynamical systems with the following properties:

(1) Occurrence of n-quasiperiodicity in a measure theoretically significant way.
(2) The system is local, and the property is determined by a low order jet.
(3) Preferably parameters are only needed in the linear part.

Below we present a solution to this problem by means of coupled Hopf families. To fix thoughts, we start with an example.

***Example 1* (Weakly coupled Hopf bifurcations).** — Consider a C^∞-system of n weakly coupled Hopf bifurcations, near the origin of $\mathbb{R}^{2n}$ given by

$$\begin{pmatrix} \dot{x}_j \\ \dot{y}_j \end{pmatrix} = \begin{pmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} - (x_j^2 + y_j^2) \begin{pmatrix} x_j \\ y_j \end{pmatrix} + O(r^4), \tag{1}$$

$1 \leqslant j \leqslant n$, where $r^2 = \sum_{j=1}^n (x_j^2 + y_j^2)$. The lower order part (the 3-jet) consists of n completely decoupled Hopf bifurcations, as already considered in [**15**]. Presently, however, we include the coupling term $O(r^4)$. Moreover, we include $2n$ parameters

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$(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$, where $\alpha \in \mathbb{R}^n_+$ is small and where β varies over any compact disc $L \subset \mathbb{R}^n_+$. In multi-polar coordinates $x_j = r_j \cos \varphi_j, y_j = r_j \sin \varphi_j$ the decoupled lower order part reads

$$\begin{aligned} \dot{\varphi}_j &= \beta_j \\ \dot{r}_j &= r_j(\alpha_j - r_j^2), \end{aligned} \tag{2}$$

$1 \leqslant j \leqslant n$. Clearly for $\alpha_j > 0$, system (2) has an n-torus attractor $r_j = \sqrt{\alpha_j}$, where the dynamics is parallel given by $\dot{\varphi}_j = \beta_j$, $1 \leqslant j \leqslant n$. Our interest is with the fate of this dynamical phenomenon upon addition of the higher order perturbation $O(r^4)$.

Below we generalize the setting of Example 1, raising a similar problem. To solve this we apply both Center Manifold Theory [**14**] and KAM Theory in the dissipative setting [**17, 2, 6, 3, 7**]. We summarize the results of our investigation. First, for small $|\alpha|$, the family of n-tori is C^ℓ-persistent for such perturbations, where the bound on $|\alpha|$ depends on ℓ. Second, the continuum of parallel dynamics persists as a Whitney smooth family of quasiperiodic attractors, foliated over a Cantor set. Projected to the (α, β)-parameter space, this Cantor foliation has positive measure, expressed in terms of a Lebesgue density point of quasiperiodicity corresponding to $\alpha = 0$. Notably, the dynamics in between generically break up due to internal resonance: upon variation of parameters the dynamics can be asymptotically periodic (e.g., phase locked) or chaotic ($n \geqslant 3$), [**20, 18, 21, 23**].

Let us briefly outline the contents of this paper. We start by generalizing the setting of Example 1, and developing an appropriate perturbation model for the application of Center Manifold and KAM Theory. We end by a general discussion, pointing towards some interesting problems regarding quasiperiodic Hopf bifurcation that occur in a subordinate way.

2. Coupled Hopf-bifurcations

2.1. Setting of the problem. — Instead of weakly coupled case (1) we here consider the more general system

$$\begin{pmatrix} \dot{x}_j \\ \dot{y}_j \end{pmatrix} = \begin{pmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} + O(r^2), \tag{3}$$

which will be subject to suitable C^3-open conditions. As in Example 1 we include dependence on the parameter vector $(\alpha, \beta) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$. We apply a standard normal form procedure to system (3), e.g. compare [**22, 4, 24, 11**]. Note that the linear part of (3) has a $\mathbb{T}^n$-symmetry. Strong resonances are excluded by requiring that

$$\sum_{j=1}^n \beta_j k_j \neq 0 \text{ whenever } 0 < \sum_{j=1}^n |k_j| \leqslant 4, \tag{4}$$

which amounts to the first open condition. Granted (4), by near identity, polynomial changes of variables this $\mathbb{T}^n$-symmetry can be pushed over the whole 3-jet, which then in appropriate multi-polar coordinates reads

$$\begin{aligned} \dot{\varphi}_j &= \beta_j - f_j(r_1^2, \dots, r_n^2) + O(r^4) \\ \dot{r}_j &= r_j(\alpha_j - g_j(r_1^2, \dots, r_n^2)) + O(r^4), \end{aligned} \tag{5}$$

$1 \leqslant j \leqslant n$. As in Example 1, we truncate the $O(r^4)$ term, so arriving at the present generalization of (2), which we explore for invariant n-tori. Therefore we expand

$$\begin{aligned} f_j &= f_{j1} r_1^2 + \cdots + f_{jn} r_n^2 \\ g_j &= g_{j1} r_1^2 + \cdots + g_{jn} r_n^2, \end{aligned}$$

with f_{ji}, g_{ji} constants, $i, j = 1, 2, \dots, n$. Invariant n-tori then are determined by n equations

$$g_{j1} r_1^2 + \cdots + g_{jn} r_n^2 = \alpha_j, \tag{6}$$

$1 \leqslant j \leqslant n$. Consider the 'action-space' $\mathbb{R}_+^n = \{r_1^2, \dots, r_n^2\}$, where the equations (6) determine n hyperplanes. Considering the $n \times n$-matrix

$$G = \left((g_{j,i})_{j,i=1}^{\ \ n} \right),$$

we impose further C^3-open conditions

$$\det G \neq 0 \text{ while } G^{-1}(\alpha) \in \mathbb{R}_+^n. \tag{7}$$

By $c^2 := G^{-1}(\alpha)$ denote the unique (transversal) intersection point of the hyperplanes. Then the equations (6) have the unique solution

$$r_1^2 = c_1^2, \dots, r_n^2 = c_n^2, \text{ with } c_j^2 = c_j^2(\alpha), \tag{8}$$

$1 \leqslant j \leqslant n$, which determines our invariant n-torus, carrying parallel dynamics.

Remark. — The open conditions (7) are trivially satisfied in Example 1, where $G = \mathrm{Id}_n$. One easily detects other concrete examples that are C^3-nearby (1).

As announced in Example 1, the problem is to study the effect of the higher order perturbaton $O(r^4)$ on this family of tori. As said before, to answer this we shall apply both Center Manifold Theory [**14**] and KAM Theory in the dissipative setting [**6, 3, 7**].

2.2. An appropriate perturbative setting. — We formulate a perturbation problem suitable for our purposes. Introducing the small parameter ε and putting $I_j = r_j - c_j$, we scale

$$\begin{aligned} \alpha_j &= \varepsilon^2 \overline{\alpha}_j \qquad & \beta_j &= \overline{\beta}_j \\ r_j &= \varepsilon \overline{r}_j & \varphi_j &= \overline{\varphi}_j, \end{aligned}$$

also writing $I_j = \varepsilon \overline{I}_j$, $1 \leqslant j \leqslant n$. This gives the estimates

$$\begin{aligned} \dot{\varphi}_j &= \overline{\beta}_j + \varepsilon O(\overline{I}) \\ \dot{\overline{I}}_j &= -2\varepsilon^2 \overline{\alpha}_j \overline{I}_j + \varepsilon^2 \overline{I}_j O(\sqrt{\overline{\alpha}} \overline{I}), \end{aligned}$$

$1 \leqslant j \leqslant n$, which are uniform for $(\overline{\alpha}, \overline{\beta}) \in K \times L$, for any given compact subsets $K, L \subseteq \mathbb{R}^n_+$. A further scaling

$$\overline{I} = \varepsilon^q \overline{\overline{I}},$$

for a fixed $q > 1$, leads to the following perturbation problem:

$$\begin{aligned} \dot{\varphi}_j &= \overline{\beta}_j + O(\varepsilon^{1+q}) \\ \dot{\overline{\overline{I}}}_j &= -2\varepsilon^2 \overline{\alpha}_j \overline{\overline{I}}_j + O(\varepsilon^{2+q}), \end{aligned} \tag{9}$$

$1 \leqslant j \leqslant n$, again with uniform estimates. Since $q > 1$, the form (9) is suitable for application of the Center Manifold Theorem [**14**], Thm. 4.1, implying the C^ℓ-persistence of the invariant n-torus for small values of ε.

To further investigate persistence of the quasiperiodic dynamics we apply dissipative KAM Theory as developed in [**6**] § § 4 and 8. In the unperturbed case

$$\begin{aligned} \dot{\varphi}_j &= \overline{\beta}_j \\ \dot{\overline{\overline{I}}}_j &= -2\varepsilon^2 \overline{\alpha}_j \overline{\overline{I}}_j, \end{aligned}$$

$1 \leqslant j \leqslant n$, we single out parameter vectors $\beta \in L$ such that for all $k \in \mathbb{Z}^n \setminus \{0\}$ Diophantine conditions

$$|\langle k, \beta \rangle| \geqslant \frac{\gamma}{|k|^\tau} \tag{10}$$

hold. Here $\tau > n - 1$ is a constant, while we choose $\gamma = c\varepsilon^q$, for an appropriate (sufficiently small) constant c, depending on K and L. These conditions define a Cantor foliation $\mathcal{C}_{\varepsilon,c} \subseteq K \times L$ for the unperturbed system. The complement $(K \times L) \setminus \mathcal{C}_{\varepsilon,c}$ has measure $O(\varepsilon^q)$ as $\varepsilon \downarrow 0$. The main result of the present paper is:

***Theorem 2* (Perturbation Theorem)**. — *Consider system (9), with parameter vectors $(\overline{\alpha}, \overline{\beta}) \in K \times L$, for given compact subsets $K, L \subseteq \mathbb{R}^n_+$. Also let $\ell \in \mathbb{N}$ be sufficiently large. Then, for $\varepsilon > 0$ and sufficiently small the (9) has the following properties:*

(1) *The unperturbed n-torus family $I = 0$ persists as a unique C^ℓ-family of hyperbolic n-torus attractors $\mathcal{T}_\varepsilon$, also depending C^ℓ on ε.*

(2) *For parameter values $(\overline{\alpha}, \overline{\beta}) \in \mathcal{C}_{\varepsilon,c}$ as described above, with c sufficiently small, the unperturbed tori persist as quasiperiodic tori inside $\mathcal{T}_\varepsilon$.*

(3) *The union of tori inside $\mathcal{T}_\varepsilon$ with non-quasiperiodic dynamics has Lebesgue measure $O(\varepsilon^q)$, as $\varepsilon \downarrow 0$, $1 \leqslant j \leqslant n$, uniformly in $(\overline{\alpha}, \overline{\beta}) \in K \times L$.*