LEONID A. BUNIMOVICH

Walks in rigid environments: symmetry and dynamics

Astérisque, tome 286 (2003), p. 231-248
<http://www.numdam.org/item?id=AST_2003__286__231_0>

Astérisque
286, 2003, p. 231–248

WALKS IN RIGID ENVIRONMENTS: SYMMETRY AND DYNAMICS

by

Leonid A. Bunimovich

Abstract. — We study dynamical systems generated by a motion of a particle in an array of scatterers distributed in a lattice. Such deterministic cellular automata are called Lorentz-type lattice gases or walks in rigid environments. It is shown that these models can be completely solved in the one-dimensional case. The corresponding regimes of motion can serve as the simple dynamical examples of diffusion, sub- and super-diffusion.

1. Introduction

Deterministic (dynamical systems) or stochastic (random processes) models are the ones which were used traditionally to model real phenomena and processes. The theory of these two types of models, purely deterministic and purely stochastic ones, is very rich and therefore the intuition on evolution of such systems is well developed. The intuition means a right expectation of what should happen in the course of evolution of some concrete system even though the rigorous mathematical analysis is usually lacking.

Such intuition is based on some explicitly solvable simple (but nontrivial) and visible examples, i.e., on the comprehensive mathematical analysis of the corresponding models. These fundamental models in the theory of stochastic processes include sequences of identically distributed independent random variables (Bernoulli shifts), a random walk, etc. In dynamical systems such fundamental models include a rotation of a circle, an algebraic toral automorphism, some billiard models, etc. Certainly, this class of completely solvable models is growing, and our intuition is essentially growing with it. I cannot resist to mention the quadratic family which now finally belongs to this class as well [**14**].

2000 ***Mathematics Subject Classification.*** — 37A50, 60K35, 82C20, 82C41.
Key words and phrases. — Deterministic walks in random environments, Lorentz lattice gases, scatterers, diffusion, sub-diffusion, super-diffusion.

This work was partially supported by NSF grant DMS-9970215.

However, dynamics of many (and actually of a majority) of real systems is neither purely stochastic nor purely deterministic but it rather has both these components. Certainly, it is the well known fact and traditional attempts to account for that is to study e.g., small random perturbations of dynamical systems or to add a small deterministic flow term (advection) to a diffusion process. Such small perturbations, while being very important to study, do not address the question on the behavior of hybrid systems with (nonsmall) deterministic and stochastic features in their evolution. In fact, in applications almost always models were chosen as stochastic ones (instead of hybrid ones) with the standard argument that each real phenomenon or process has infinitely many features neglected by any model and therefore it is, in fact, a random process.

There are large areas like e.g., operations research, logistics, etc., which still completely belong to the probability theory while already the first applications of the dynamical systems methods allowed to achieve very encouraging results by essentially increasing production rates of certain production lines [**2**].

Another class of hybrid systems goes back to the classical Lorentz gas. Recall that in the Lorentz gas (light) point, noninteracting between themselves, particles move by inertia in an array of immovable scatterers and collide with scatterers elastically. It is a dynamical system which can be reduced to Sinai billiard. This system has been comprehensively studied and until now it is the only one nontrivial system for which time irreversible macroscopic dynamics (governed by the diffusion equation) has been rigorously derived from the time reversible microscopic dynamics (governed by Newton equations). It is transparent that this result has been obtained only for periodic configurations of scatterers (under the condition that a free path of the point particle is bounded, see details in [**6**]).

The very interesting mathematically and important problem for various applications is to study this system in case when the scatterers are distributed randomly. It seems, at the first sight, that this problem should follow from the one with periodic distribution of scatterers because of some additional "self-averaging" generated by a random distribution of scatterers. Indeed, it seems that such "self-averaging" should just improve stochastic properties of the corresponding dynamical system with periodically placed scatterers. However, this idea is totally wrong. In fact, in the Lorentz gas with randomly distributed scatterers we encounter a hybrid system, which has both deterministic and stochastic features. (Certainly, the Lorentz gas with randomly distributed scatterers can be described as purely deterministic (dynamical) system. However, it does not make this system to be deterministic, as well as the representation of a stationary random process a shift in the space of its realizations does not transform this stochastic process into a deterministic one.)

If an interesting and important system does not allow a comprehensive analysis then it is natural to consider some simpler model which retains (some) principal features of this system. Such simplified Lorentz gas model has been introduced in [**18**]. In this model scatterers (usually of two different types, e.g., left and right mirrors aligned along the diagonals of the square lattice) are randomly distributed on vertices of the square lattice. The point particle moves with unit speed along the bonds of this lattice and get reflected by the scatterers. These systems were naturally called Lorentz Lattice Gases (LLG). It is worthwhile to mention that this model is the generalization of another classical model in nonequilibrium statistical mechanics, which is called the (Ehrenfests') Wind-Tree model. In the Wind-Tree model a (light) point particle moves in an array of randomly distributed scatterers, which are identical rhombuses with parallel diagonals. The particle moves parallel to one of the diagonals of rhombuses and therefore after (elastic) reflection from the boundary of some scatterer, its velocity becomes parallel to another diagonal of the rhombuses and so on.

The Lorentz Lattice Gases belong to the class of systems which can be naturally called Deterministic Walks in Random Environments (DWRE). Indeed the dynamics of these systems is generated by deterministic motion of the particle, where both the free motion and reflections from the boundary of scatterers are deterministic, while distribution of scatterers is random.

It occurred that the Lorentz Lattice Gases were studied (without using this name) in lots of applications, e.g., in material science, superconductivity, chemical kinetics, information transmission and especially in the theoretical computer science. All these studies were exclusively numerical and these systems were included in the class of systems which are conventionally called "complex systems" (and are often discussed in the journal with the same name).

In fact, in many applications there were considered so called flipping LLG, where the moving particle has impact on an environment as well. Formally dynamics of such models is defined by the rule that after reflection of the moving particle from a scatterer this scatterer instantly changes its type. Therefore in flipping LLG there is also a dynamics of an environment formed by the configuration of scatterers. Hence for such models it makes sense to consider dynamics of many particles moving along the bonds of a lattice rather than of a single one. Indeed, even though the moving particles do not interact directly they do, in fact, interact via changing the environment to each other. It allows to account for an "information exchange" between particles (signals, etc.) and environment (neurons, etc.), see. e.g., [**1, 7, 9, 10, 12**].

From the mathematical point of view all these models are dynamical systems. In fact, they belong to the class of deterministic cellular automata. However, this formal observation does not help much in studying these systems. In fact, it occurred that the much more productive approach is to consider all these models as Deterministic Walks in Random Environments. (To make clear distinction with purely stochastic models of this kind we mention that in the last ones a scatterer after colliding with particle "flips a coin" to decide whether it should change its type.)

In the studies of DWRE the important role is played both by the structure of a lattice where particles move (which could be e.g., the square, triangular, cubic, random, etc. lattice) and by the types of scatterers considered (e.g., there are 4^4 types of scatterers in a square lattice). It is not surprising, of course, because a lattice defines a configuration space and the types of scatterers (together with a lattice) define the dynamics (equations of motion).

The great majority of papers on DWRE are numerical. There are as well quite a few mathematical results on dynamics of DWRE. They usually use some specific features of the given model, which allow sometimes to come up with complete solution. For instance, it is possible to reduce a (purely deterministic) problem to a (purely probabilistic) percolation problem on some graph [**4**]. (It is worthwhile to mention that such graph is defined not only by the lattice but by the types of scatterers as well.) Sometimes it was possible to completely solve the problem by constructing some peculiar class of solutions and by proving that no other solutions exist (see e.g., [**5**]). However, in most cases the results were rather counterintuitive. Actually, in almost all cases when dealing with the hybrid (neither purely deterministic, nor purely stochastic) systems the authors confessed that they obtained results different from what they expected.

This situation clearly calls for some kind of a general view at these systems, especially the one which would allow to integrate the studies of DWRE in fixed and in evolving (e.g., flipping) environments. The corresponding approach has been developed in [**3**] where these two classes of DWRE, were integrated into one class of dynamical systems called Walks in Rigid Environments (WRE). (Observe that R in DWRE refers to "random," while in WRE it refers to "rigid").

WRE is also a dynamical system generated by motion of point particles in some graph (e.g., in a lattice). For the sake of simplicity we will consider here only one-particle systems. Some scatterers are randomly distributed along the vertices of this graph. (Again for the sake of simplicity we assume that the scatterers are distributed independently, even though one may assume that they interact via some potential.)

The crucial feature of WRE is the new parameter r which is called a rigidity of an environment. The rigidity determines how many times the particle must collide with the given scatterer in order to change its type. In the other words, the scatterer at a given site changes its type at the moment after the rth visit of the moving particle to this site. It is easy to see that the LLGs with fixed environment correspond to the case $r = \infty$, while the LLGs with flipping environment correspond to the case $r = 1$. Thus the two studied so far classes of LLG form, in fact, two extreme sub-classes of WRE.

Besides the introduction of Walks in Rigid Environments allowed to move rigorous studies of LLG to another level and to address the central problem of the theory of such systems which is the diffusion problem. Until [**3**] the mathematical papers on Deterministic Walks in Random Environments usually addressed the problem whether a typical path of a particle is bounded or unbounded. However, the most important question which one can ask about evolution of a system generated by a motion of some