Astérisque

ALAIN CHENCINER

Perverse solutions of the planar n-body problem

Astérisque, tome 286 (2003), p. 249-256

<http://www.numdam.org/item?id=AST_2003__286__249_0>

Astérisque
286, 2003, p. 249–256

PERVERSE SOLUTIONS OF THE PLANAR n-BODY PROBLEM

by

Alain Chenciner

To Jacob, some questions for his 60th anniversary

Abstract. — The perverse solutions of the n-body problem are the solutions which satisfy the equations of motion for at least two distinct systems of masses. I contribute with some simple remarks concerning their existence, a question which curiously seems to be new.

Let $X(t) = \big(\vec{r}_1(t), \vec{r}_2(t), \ldots, \vec{r}_n(t)\big)$ be a solution of the n-body problem with newtonian potential and masses $m_1, m_2, \ldots, m_n$. We ask the following questions:

Question 1. — *Does there exist another system of masses, $(m'_1, m'_2, \ldots, m'_n)$, for which $X(t)$ is still a solution ?*

Question 2. — *Same as question 1 but insisting that the sum $M = \sum_{i=1}^{n} m_i$ of the masses and the center of mass $\vec{r}_G = (1/M)\sum_{i=1}^{n} m_i\vec{r}_i$ do not change.*

Definition. — *If the answer to the first (resp. second) question is yes, we shall say $X(t)$ is a* perverse *(resp.* truly perverse*) solution and the allowed systems of masses will be called* admissible.

Remark. — If the inverse problem raised by Question 1 may seem very natural, Question 2 needs some motivation. The possible existence of *choreographies* whose masses are not all equal is at the origin of the notion of perverse solution. Recall that a planar choreography is a periodic solution $\mathcal{C}(t) = \big(q(t+T/n), \ldots, q(t+(n-1)T/n), q(t+T) = q(t)\big)$ of the n-body problem such that all n bodies follow the same closed plane curve $q(t)$ with equal time spacing ([**S1, S2, CGMS**]). It is noticed in [**C**] that if a choreography exists whose masses are not all equal, it is a truly perverse choreography: by replacing each mass by the mean mass M/n we obtain new admissible masses, while keeping the center of mass and total mass unchanged.

2000 ***Mathematics Subject Classification.*** — 70F10.
Key words and phrases. — n-body problem, homographic solutions.

In the sequel, we shall consider only the planar problem. We shall identify the plane of motion with the complex plane $\mathbb{C}$, hence the positions $\vec{r}_G, \vec{r}_i$, $i = 1, \dots, n$, with complex numbers z_G, z_i, $i = 1, \dots, n$, and $X(t)$ with an element of $\mathbb{C}^n$. We shall use the following notations (we always assume that $z_i \neq z_j$):

$$\begin{cases} z_{ij} & = z_i - z_j, \quad a_{ij} = \dfrac{z_{ij}}{|z_{ij}|^3} \text{ if } i \neq j,\ a_{ii} = 0, \quad m = (m_1, m_2, \dots, m_n), \\ \mathcal{A}_0 & = \big(z_{ij}\big)_{1 \leqslant i,j \leqslant n}, \quad \mathcal{A} = \big(a_{ij}\big)_{1 \leqslant i,j \leqslant n} \end{cases}$$

We shall identify a matrix as $\mathcal{A}_0$ or $\mathcal{A}$ with a linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$. This will allow it to act on the vector m. The definition of the center of mass may be rewritten

$$\sum_{j=1}^{n} m_j z_{ij} = M(z_i - z_G), \quad M = \sum_{j=1}^{n} m_j,$$

that is

$$\mathcal{A}_0(t)m = M\big(X(t) - z_G(t)(1, \dots, 1)\big),$$

and the equations of motion in a galilean frame are

$$\forall\, t, \forall\, i,\ \ddot{z}_i(t) = -\sum_{j \neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^3}, \quad \text{that is} \quad \mathcal{A}(t)m = -\ddot{X}(t).$$

Hence, if another set $m'_1, m'_2, \dots, m'_n$ of masses admits the same solution $X(t)$, the difference

$$\mu = m - m' = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n$$

is a real non-zero vector in the kernel of any of the complex matrices $\mathcal{A}(t)$. If, moreover, M and $z_G(t)$ are the same for the two sets of masses, μ is also in the kernel of any of the matrices $\mathcal{A}_0(t)$. It will be important to remember that $\mathcal{A}_0$ and $\mathcal{A}$ are antisymmetric (${}^t\mathcal{A}_0 = -\mathcal{A}_0$, ${}^t\mathcal{A} = -\mathcal{A}$). This will cause the parity of n to play a role. We start with the obvious

Proposition 1. — *If $n = 2$, no solution is perverse. In other words, any planar solution of the 2-body problem determines the masses.*

Proof. — If $n = 2$, the matrix $\mathcal{A}(t)$ is of maximal rank whenever it is defined, that is provided $z_{12}(t) \neq 0$. □

As soon as $n \geqslant 3$, perverse solutions do exist, as shown by the following "trivial" examples (thanks to Reinhart Schäfke for proposing immediately the example of an equilateral triangle rotating around a fourth body):

Example 1. — $X(t) = \big(re^{i\omega t}, re^{i\omega t + \frac{2i\pi}{n-1}}, \dots, re^{i\omega t + (n-2)\frac{2i\pi}{n-1}}, 0\big)$ is a relative equilibrium solution with n masses $(m_1, m_1, \dots, m_1, m_0)$ if and only if the following "Kepler-like" condition is satisfied:

$$r^3\omega^2 = \frac{U_n}{I_n} = m_0 + \frac{m_1}{n-1} \sum_{1 \leqslant j < k \leqslant n-1} \frac{1}{|z_{jk}|}.$$

In the above formula,

$$U_n = m_1 m_0 (n-1) + m_1^2 \sum_{1\leqslant j<k\leqslant n-1} \frac{1}{|z_{jk}|} \quad \text{and} \quad I_n = m_1(n-1)$$

stand respectively for the potential and the moment of inertia with respect to the center of mass, of the configuration normalized by $|z_{in}| = 1$ if $1 \leqslant i \leqslant n-1$. This leaves a one parameter family of admissible sets of masses. Moreover, for the regular $(n-1)$-gon inscribed in the unit circle, we have

$$\sum_{1\leqslant j<k\leqslant n-1} \frac{1}{|z_{jk}|} = \frac{n-1}{2}\Big(\frac{1}{2\sin\frac{\pi}{n-1}} + \frac{1}{2\sin\frac{2\pi}{n-1}} + \cdots + \frac{1}{2\sin\frac{(n-2)\pi}{n-1}}\Big) = (n-1)^2(\delta_{n-1}+1),$$

where we have set

$$\delta_n = -1 + \frac{1}{4n}\sum_{l=1}^{n-1} \frac{1}{\sin\frac{\pi l}{n}}.$$

Hence,

$$r^3\omega^2 = m_0 + (n-1)m_1(\delta_{n-1}+1) = M + (n-1)m_1\delta_{n-1}.$$

Provided δ_{n-1} is different from 0, the right hand side of the above formula is a linear form in the masses which is linearly independent of the total mass $M = m_0 + m_1(n-1)$.

But δ_{n-1} is strictly negative if $n-1 \leqslant 472$ and strictly positive if $n-1 \geqslant 473$ (see [**MS**]; the first occurence of the magic number 472 seems to be in [**M**]). It follows that M may be chosen as a natural parameter of the set of admissible masses. In particular, these examples are *perverse* but not *truly perverse*.

Remark. — For non-newtonian potentials of the form $1/r^{2\beta}$, $\beta \neq 1/2$, the analogue of δ_n becomes

$$\delta_n = -1 + \frac{1}{2^{2\beta+1}n}\sum_{l=1}^{n-1} \frac{1}{(\sin\frac{\pi l}{n})^{2\beta}},$$

and may become zero for some value of β (see [**BCS**]).

Example 2. — Similar to Example 1 are the relative equilibrium solutions whose configuration is made of one central mass m_0 and k regular homothetic n-gons, the masses in the j-th polygon being all equal to m_j, for $j = 1, \dots, k$. In this case, the equations insuring relative equilibrium motion may be put in the form (see [**BE**] or [**BCS**]):

$$\rho_j^3\omega^2 = m_0 + \sum_{s=1}^{k} m_s H_n(\rho_s/\rho_j), \quad j = 1, \dots, k,$$

where ρ_j is the radius of the j-th polygon and

$$H_n(x) = \sum_{l=1}^{n^*(x)} \frac{1 - x\cos\frac{2\pi l}{n}}{(1 + x^2 - 2x\cos\frac{2\pi l}{n})^{3/2}}, \quad n^*(x) = \begin{cases} n & \text{if } x \neq 1, \\ n-1 & \text{if } x = 1. \end{cases}$$

In the "generic" case, such solutions will be perverse and not truly perverse. But, as soon as $k \geqslant 3$, one gets truly perverse solutions for special choices of the radii ρ_j and the integer n (see the last section).

When $n = 3$, the situation is still easy to deal with, thanks to Albouy and Moeckel [**AM**].

Proposition 2. — *The perverse solutions of the planar 3-body problem are exactly the collinear homographic solutions. The center of mass is the same for all admissible sets of masses, but not the total mass, which is a natural parameter for such sets. In particular, truly perverse solutions do not exist.*

Proof. — If $n = 3$, the matrix $\mathcal{A}(t)$ is of rank 2 as soon as the configuration is not a triple collision. The existence of a fixed non-zero real vector μ in the kernel of $\mathcal{A}(t)$ implies immediately that the three bodies stay collinear, with a fixed configuration up to similarity. This implies that the motion is homographic. Moreover, the center of mass is dynamically defined as the unique common focus of the similar conics described by the bodies in a galilean frame where the center of mass corresponding to one admissible choice of masses is fixed.

Conversely, each collinear homographic solution of the 3-body problem is perverse: this is a direct consequence of Theorem 2 and Proposition 4 of [**AM**] which, together, say that the set of masses for which a given configuration of three bodies is central is of dimension 2 and may be parametrized by the "multiplier" λ (which is determined by the equation $\ddot{X} = -\lambda X$ as soon as the homographic solution X is given) and the total mass M. To finish the proof, it remains to recall that the center of mass of such a 3-body configuration does not depend on the choice of masses for which it is central (see [**AM**] where this observation is attributed to C. Marchal). □

The case $n = 4$. — The determinant of the antisymmetric 4×4 matrix $\mathcal{A}$ is equal to the square of the Pfaffian (if we extend the notation K_4 of [**AM**] to the complex domain, $P = K_4/2$)

$$P(z_1, z_2, z_3, z_4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Hence, if a solution of the 4-body problem admits two different sets of masses, its configuration must satisfy $P\big(z_1(t), z_2(t), z_3(t), z_4(t)\big) = 0$ at each instant t. As in [**AM**], but in the complex setting, let us use the following notations :

$$A = z_{12}z_{34}, \; B = z_{13}z_{24}, \; C = z_{14}z_{23}.$$

The above condition becomes

$$P = \frac{A}{|A|^3} - \frac{B}{|B|^3} + \frac{C}{|C|^3} \equiv 0.$$