Astérisque

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Astérisque, tome 286 (2003), p. 257-308
<http://www.numdam.org/item?id=AST_2003__286__257_0>

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286, 2003, p. 257–308

CHAOS VERSUS RENORMALIZATION AT QUADRATIC S-UNIMODAL MISIURWICZ BIFURCATIONS

by

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Abstract. — We study C^3 families of unimodal maps of the interval with negative Schwarzian derivative and quadratic critical point, transversally unfolding Misiurewicz bifurcations, and for these families we prove that existence of an absolutely continuous invariant probability measure ("chaos") and existence of a renormalization are prevalent in measure along the parameter. Moreover, the method also shows that existence of a renormalization is dense and chaos occurs with positive measure.

1. Introduction

The *quadratic family*

$$\begin{array}{rcl} f_a : [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & 4ax(1-x) \end{array}, \quad a \in [0,1],$$

is the simplest model that shows the complexity arising in nonlinear dynamical systems. For a fixed value of the parameter a, supposed to vary along the interval $[0,1]$, one is interested to follow the behavior of iterates x_0, $x_1 = f_a(x_0)$, $x_2 = f_a(x_1)$, ..., in other words of orbits

$$\mathcal{O}(x_0) = \{f_a^n(x_0)\}_{n \geqslant 0}$$

starting at a point x_0. The set $\omega(x_0)$ of accumulation points of $\mathcal{O}(x_0)$ gives a clue of the asymptotic behavior of the orbit of x_0, and is called the *ω-limit set of* x_0. It turns out ([**7**]) that "typical" starting points $x_0 \in [0,1]$ have equal ω-limit sets. This could be stated as follows: *for each $a \in [0,1]$, there is a set $A = A_a$ such that $\omega(x_0) = A$ for Lebesgue almost every $x_0 \in [0,1]$*. Moreover, there are only three types of sets which A_a could be: (i) a *periodic orbit*, i.e. a set $\{p_0, p_1, \dots, p_{k-1}\}$ such that $f_a(p_0) = p_1$, $f_a(p_1) = p_2$, ..., $f_a(p_{k-1}) = p_0$; (ii) a *periodic collection of pairwise disjoint intervals*

2000 ***Mathematics Subject Classification.*** — 37E05, 26A18, 37E20.
Key words and phrases. — Unimodal maps, chaos, absolutely continuous invariant measure, renormalization, generalized renormalization, bifurcation.

$\{I_0, I_1, \ldots, I_{k-1}\}$ where $f_a(I_0) = I_1$, $f_a(I_1) = I_2$, $\ldots$, $f_a(I_k) = I_0$; or (iii) a *Cantor set* (i.e. a perfect and totally disconnected compact set) of zero Lebesgue measure.

The striking alternation of behavior of f_a has been revealed and proved along the last three decades. Among others, we know that: parameters for which the typical ω-limit set is a periodic orbit are dense (and contain intervals, implying also positive Lebesgue measure) ([**3**], [**8**]); parameters for which the typical ω-limit set is a collection of intervals have positive measure (following [**4**]); and parameters for which the typical ω-limit set is a Cantor set have zero Lebesgue measure ([**10**]).

Among parameters with a cycle of intervals as its typical ω-limit set, with total Lebesgue measure ([**9**], [**12**]) we find those for which there is an absolutely continuous (with respect to Lebesgue) f_a-invariant probability measure. In this case f_a is said to be *chaotic*, although more intuitive and not exactly equivalent definitions of "chaos" are available. This definition supplies at least some statistical properties for the mean growth of derivatives along orbits and imply some dynamical structure on the configuration space.

On the other hand, parameters where the typical ω-limit set is a non-hyperbolic periodic orbit are rare in measure. In other words, hyperbolicity is prevalent in measure for these parameters. Putting things altogether, we conclude that for almost all $a \in [0,1]$, the dynamics of f_a is *either hyperbolic or chaotic*.

A largely used concept in one-dimensional dynamics is the idea of renormalization. We say that f_a is *renormalizable* if there is a collection of pairwise disjoint intervals $\{I_0, I_1, \ldots, I_{k-1}\}$ properly contained in $[0,1]$ such that (i) the critical point $\frac{1}{2}$ of f_a belongs to, say, I_{k-1}; (ii) $f_a(I_{k-1}) \subset I_0$ and $f_a(\partial I_{k-1}) \subset \partial I_0$; (iii) $f_a : I_i \to I_{i+1}$ is a diffeomorphism for all $i = 0, \ldots, k-2$. In particular, if we call $I = I_{k-1}$, then the function $f_a^k|I$ resembles in many ways the general aspect of a quadratic function in $[0,1]$, since $f_a^k(I) \subset I$, $f_a^k(\partial I) \subset \partial I$ and $f_a^k|I$ has a single (quadratic) critical point (equal to $\frac{1}{2}$). By an affine rescaling a new function $g : [0,1] \to [0,1]$ could be defined, but in general we may not expect g to be quadratic.

Renormalization is a kind of reduction tool. For example, the behavior of typical orbits is completely determined by the restriction $f_a^k|I$, since we know (see [**13**] and references therein) that for Lebesgue almost every $x \in [0,1]$ there is $n = n(x)$ such that $f_a^n(x) \in I$. All subsequent iterates must remain inside the cycle from this iterate on, because of the invariance properties stated above. This suggests that no complete knowledge of the quadratic family could be achieved without the understanding of a larger class of functions which contains in particular the ones generated *via* renormalization. For this class, it would be desirable some qualitative dynamical similarity with quadratic functions, not only for technical reasons (proves with recursive arguments) but also for the sake of some universality in the conclusions.

In [**3**] and [**8**] (denseness of hyperbolicity), [**9**] joint with [**12**] (measure prevalence of chaos) and [**10**] (rareness of Cantor ω-limit sets), this larger class of functions to which the quadratic functions belong (and which is invariant under renormalization) is composed by all analytic functions f which are holomorphically extendible to a neighborhood U of $[0,1]$ in the complex plane, such that $f(U)$ contains the closure of U and f is a double branched covering between U and $f(U)$. Recently ([**1**]) there have been considered the case of real analytic functions, but even so some main arguments are based on constructions developed in the complex plane.

Among the results mentioned for the quadratic family, the positive measure of chaotic parameters, proved for the first time in [**4**], is the only one which has been stated for C^2 families (see for example [**16**] or [**13**] and references therein). The present work is an attempt to provide techniques restricted to the real setting, weakening smoothness considerably, in order to state results that go in the same direction as the ones of the previous paragraph. Unfortunately the extent of the conclusions cannot be as complete as the ones already proved for the quadratic family. The main reason is that our statements are of a local nature, that is, they are valid only for parameters in small intervals around some bifurcation values. This does not allow us to go beyond the first renormalization, where full families appear.

Here we deal with C^3 *unimodal* interval maps f, that is those with a single turning point c, with the (classical) additional hypothesis that the Schwarzian derivative

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

defined for all $x \neq c$, is non-positive. These functions will be called *S-unimodal.* From this hypothesis some *a priori* conclusions can be derived. For example, there is at most one periodic attractor and if it does exist then it must attract the critical orbit $\mathcal{O}(c)$ ([**15**]). Moreover, distortion of derivatives for powers of f can be uniformly controlled (see statements in [**13**]). This comes from two facts: first, if a diffeomorphism defined in an interval I has non-positive Schwarzian derivative, the ratio between its derivatives evaluated at two points can be bounded by a constant which depends only on the proportion between their mutual distance and their distance to the boundary of I, *but not on the diffeomorphism.* Second, powers of f have also non-positive Schwarzian derivatives, hence distortion bounds may be obtained whenever $f^n|I$ is a diffeomorphism for some I, independently of n.

To make clear the results we want to state below, it is convenient to relate renormalization with the classification of functions into three types we have made above, which are still valid for the larger class we are considering now (see [**7**]). First, we observe that if f is renormalizable then there is an interval $I^{(1)}$ containing the critical point and a number k_1 such that $f^{k_1}|I^{(1)}$ is a unimodal function. It may be that this function is also renormalizable, and in this case we say that f is (at least) twice renormalizable. We can take the maximum chain of renormalization intervals ordered by (proper) inclusion

$$[0,1] = I^{(0)} \supset I^{(1)} \supset I^{(2)} \ldots.$$

If this chain has size $N+1$ then we say that f is N times renormalizable, and if its size is not finite ($N=\infty$) then we say that f is *infinitely renormalizable*. The case where the size is equal to 1 is called *non-renormalizable*.

It turns out that f is infinitely renormalizable if and only if typical points have a Cantor set as its ω-limit set ([**13**]). If f is N times renormalizable, its typical ω-limit set is determined by the N-th renormalization $g=f^{k_N}|I^{(N)}$. If g has an attracting fixed point, the ω-limit set is a periodic orbit, otherwise a collection of intervals. Here we are using the fact that if g had an attracting point of period greater or equal than two then g would be renormalizable, characterizing a contradiction.

We say that f is *Misiurewicz* if the critical point c is not recurrent, i.e. $c\notin\omega(c)$. It may happen that $\omega(c)$ is an attracting periodic orbit. If not, then $f(c)$ belongs to a hyperbolic invariant compact set $\Lambda=\Lambda_f$. From hyperbolic theory, we know that for g sufficiently near f (in the C^1 topology), there is a g-hyperbolic invariant compact set Λ_g such that $f|\Lambda_f$ and $g|\Lambda_g$ are conjugated by $h_g:\Lambda\to\Lambda_g$. The function $g\mapsto\Lambda_g$ is in fact C^1 and is called the *hyperbolic continuation* of Λ. Now we embed f in a C^3 family $(f_a)_a$, where $f_0=f$, and call w the point belonging to Λ such that $w=f(c)$. As a varies, w has its continuation $w_a=h_{f_a}(w)$ and the critical point c has its continuation c_a, which is well defined by the Implicit Function Theorem, using that c is quadratic. We will say that $(f_a)_a$ is *transversal at* $a=0$ if

$$\frac{d}{da}\left(f_a(c_a)-w_a\right)\neq 0.$$

Without loss of generality, we assume $c_a\equiv c$ and $\dfrac{d}{da}(f_a(c)-w_a)>0$.

Theorem 1.1. — *Let $f:[0,1]\to[0,1]$ be a C^3 S-unimodal non-renormalizable Misiurewicz function, without periodic attractors. Let $(f_a)_a$ be a C^3 family with $f_0=f$, transversal at $a=0$. Then there is $\varepsilon>0$ such that*

(1) *for almost all $a\in[-\varepsilon,\varepsilon]$, f_a is chaotic or renormalizable;*

(2) *parameters for which f_a is renormalizable constitute a countable union of closed intervals which is dense in $[-\varepsilon,\varepsilon]$;*

(3) *parameters for which f_a is at the same time non-renormalizable and Misiurewicz have zero Lebesgue measure in $[-\varepsilon,\varepsilon]$.*

All items of Theorem 1.1 are new for non-analytic families (the third item is analogous to the statements in [**14**])

As a corollary of the method, we are also able to show that parameters for which f_a is chaotic have positive Lebesgue measure in $[-\varepsilon,\varepsilon]$, assertion which has already been proved, even in more generality, for C^2 families (see [**16**] and [**13**], Chap.V, Section 6; in fact, they prove that the relative measure goes to one at the bifurcation value). The techniques, however, go in a totally different direction, since they work with exclusion of "bad" parameters (which in general include everyone for which there is a renormalization), showing then that the remaining ones have positive measure