Astérisque

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Astérisque, tome 286 (2003), p. 25-62
<http://www.numdam.org/item?id=AST_2003__286__25_0>

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286, 2003, p. 25–62

RANDOM PERTURBATIONS OF NONUNIFORMLY EXPANDING MAPS

by

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Abstract. — We give both sufficient conditions and necessary conditions for the stochastic stability of nonuniformly expanding maps either with or without critical sets. We also show that the number of probability measures describing the statistical asymptotic behaviour of random orbits is bounded by the number of SRB measures if the noise level is small enough. As an application of these results we prove the stochastic stability of certain classes of nonuniformly expanding maps introduced in [**Vi1**] and [**ABV**].

1. Introduction

Dynamical systems theory has, among its main goals, the description of the typical behaviour of orbits as time goes to infinity, and understanding how this behaviour is modified under small perturbations of the system. This work refers to the study of the latter problem from a probabilistic point of view.

Given a map f from a manifold M into itself, let $(x_n)_{n\geqslant 1}$ be the orbit of a given point $x_0 \in M$, that is $x_{n+1} = f(x_n)$ for every $n \geqslant 1$. Consider the sequence of time averages of Dirac measures δ_{x_j} along the orbit of x_0 from time 0 to n. A special interest lies on the study of the convergence of such time averages for a "large" set of points $x_0 \in M$ and the properties of their limit measures. In this direction, we refer the work of Sinai [**Si**] for Anosov diffeomorphisms, later extended by Ruelle and Bowen [**BR, Ru**] for Axiom A diffeomorphisms and flows. In the context of systems with no uniform hyperbolic structure Jakobson [**Ja**] proved the existence of such measures for certain quadratic transformations of the interval exhibiting chaotic behaviour.

2000 ***Mathematics Subject Classification.*** — 37A05, 37A50, 37C40, 37D25.
Key words and phrases. — Random perturbations, physical measures, stochastic stability, non-uniform expansion.

Work partially supported by FCT through CMUP.

Another important contribution on this subject was given by Benedicks and Young [**BY1**], based on the previous work of Benedicks and Carleson [**BC1, BC2**], where this kind of measures were constructed for Hénon two dimensional maps exhibiting strange attractors. The recent work of Alves, Bonatti and Viana [**ABV**] shows that such measures exist in great generality for systems exhibiting some nonuniformly expanding behaviour.

The notion of stability that most concerns us can be formulated in the following way. Assume that, instead of time averages of Dirac measures supported on the iterates of $x_0 \in M$, we consider time averages of Dirac measures δ_{x_j}, where at each iteration we take x_{j+1} close to $f(x_j)$ with a controlled error. One is interested in studying the existence of limit measures for these time averages and their relation to the analogous ones for unperturbed orbits, that is, the stochastic stability of the initial system.

Systems with some uniformly hyperbolic structure are quite well understood and stability results have been established in general by Kifer and Young; see [**Ki1, Ki2**] and [**Yo**]. The knowledge of the stochastic behaviour of systems that do not exhibit such uniform expansion/contraction is still very incomplete. Important results on this subject were obtained by Katok, Kifer [**KK**], Benedicks, Young [**BY1**], Baladi and Viana [**BV**] for certain quadratic maps of the interval. Another important contribution is the announced work of Benedicks and Viana for Hénon-like strange attractors. As far as we know these are the only results of this type for systems with no uniform expanding behaviour.

In this work we present both sufficient conditions and necessary conditions for the stochastic stability of nonuniformly expanding dynamical systems. As an application of these results we prove that the classes of nonuniformly expanding maps introduced in [**Vi1**] and [**ABV**] are stochastically stable.

1.1. Statement of results. — Let $f : M \to M$ be a smooth map defined on a compact riemannian manifold M. We fix some normalized riemannian volume form m on M that we call *Lebesgue measure*.

Given μ an f-invariant Borel probability measure on M, we say that μ is an *SRB measure* if, for a positive Lebesgue measure set of points $x \in M$, the averaged sequence of Dirac measures along the orbit $(f^n(x))_{n\geqslant 0}$ converges in the weak* topology to μ, that is,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi\big(f^n(x)\big) = \int \varphi\, d\mu \tag{1}$$

for every continuous map $\varphi : M \to \mathbb{R}$. We define the *basin* of μ as the set of those points x in M for which (1) holds for all continuous φ. The maps to be considered in this work will only have a finite number of SRB measures whose basins cover the whole manifold M, up to a set of zero Lebesgue measure.

We are interested in studying random perturbations of the map f. For that, we take a continuous map

$$\begin{aligned} \Phi : T &\longrightarrow C^2(M, M) \\ t &\longmapsto f_t \end{aligned}$$

from a metric space T into the space of C^2 maps from M to M, with $f = f_{t^*}$ for some fixed $t^* \in T$. Given $x \in M$ we call the sequence $\big(f_{\underline{t}}^n(x)\big)_{n\geqslant 1}$ a *random orbit* of x, where $\underline{t}$ denotes an element $(t_1, t_2, t_3, \dots)$ in the product space $T^{\mathbb{N}}$ and

$$f_{\underline{t}}^n = f_{t_n} \circ \cdots \circ f_{t_1} \quad \text{for } n \geqslant 1.$$

We also take a family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on T such that $(\operatorname{supp}\theta_\varepsilon)_{\varepsilon>0}$ is a nested family of connected compact sets and $\operatorname{supp}\theta_\varepsilon \to \{t^*\}$ when $\varepsilon \to 0$. We will also assume some quite general nondegeneracy conditions on Φ and $(\theta_\varepsilon)_{\varepsilon>0}$ (see the beginning of Section 3) and refer to $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ as a *random perturbation* of f.

In the context of random perturbations of a map we say that a Borel probability measure μ^ε on M is *physical* if for a positive Lebesgue measure set of points $x \in M$, the averaged sequence of Dirac probability measures $\delta_{f_{\underline{t}}^n(x)}$ along random orbits $\big(f_{\underline{t}}^n(x)\big)_{n\geqslant 0}$ converges in the weak* topology to μ^ε for $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. That is,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi\big(f_{\underline{t}}^n(x)\big) = \int \varphi\, d\mu^\varepsilon \quad \text{for all continuous } \varphi\colon M \to \mathbb{R} \tag{2}$$

and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. We denote the set of points $x \in M$ for which (2) holds by $B(\mu^\varepsilon)$ and call it the *basin of* μ^ε. The map $f\colon M \to M$ is said to be *stochastically stable* if the weak* accumulation points (when $\varepsilon > 0$ goes to zero) of the physical probability measures of f are convex linear combinations of the (finitely many) SRB measures of f.

1.1.1. Local diffeomorphisms. — Let $f : M \to M$ be a C^2 local diffeomorphism of the manifold M. We say that f is *nonuniformly expanding* if there is some constant $c > 0$ for which

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leqslant -c < 0 \tag{3}$$

for Lebesgue almost every $x \in M$. It was proved in [**ABV**] that for a nonuniformly expanding local diffeomorphism f the following holds:

(P) *There is a finite number of ergodic absolutely continuous* (SRB) *f-invariant probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure subset of M. Moreover, every absolutely continuous f-invariant probability measure μ may be written as a convex linear combination of $\mu_1, \dots, \mu_p$: there are real numbers $w_1, \dots, w_p \geqslant 0$ with $w_1 + \cdots + w_p = 1$ for which $\mu = w_1\mu_1 + \cdots + w_p\mu_p$.*

The proof of the previous result was based on the existence of α-hyperbolic times for the points in M: given $0 < \alpha < 1$, we say that $n \in \mathbb{Z}^+$ is a *α-hyperbolic time* for the point $x \in M$ if

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leqslant \alpha^k \quad \text{for every} \quad 1 \leqslant k \leqslant n. \tag{4}$$

The existence of (a positive frequency of) α-hyperbolic times for points $x \in M$ is a consequence of the hypothesis of nonuniform expansion of the map f and permits us to define a map $h : M \to \mathbb{Z}^+$ giving the first hyperbolic time for m almost every $x \in M$.

In the context of random perturbations of a nonuniformly expanding map we are also able to prove a result on the finitness of physical measures.

Theorem A. — *Let $f\colon M \to M$ be a C^2 nonuniformly expanding local diffeomorphism. If $\varepsilon > 0$ is sufficiently small, then there are physical measures $\mu_1^\varepsilon, \dots, \mu_\ell^\varepsilon$ (with ℓ not depending on ε) such that:*

(1) *for each $x \in M$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$, the average of Dirac measures $\delta_{f_{\underline{t}}^n(x)}$ converges in the weak* topology to some μ_i^ε with $1 \leqslant i \leqslant \ell$;*

(2) *for each $1 \leqslant i \leqslant \ell$ we have*

$$\mu_i^\varepsilon = w^*\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \int \left(f_{\underline{t}}^j\right)_* \left(m \| B(\mu_i^\varepsilon)\right) d\theta_\varepsilon^{\mathbb{N}}(\underline{t}),$$

where $m\|B(\mu_i^\varepsilon)$ is the normalization of the Lebesgue measure restricted to $B(\mu_i^\varepsilon)$;

(3) *if f is topologically transitive, then $\ell = 1$.*

We say that the map f is *nonuniformly expanding for random orbits* if there is some constant $c > 0$ such that for $\varepsilon > 0$ small enough

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| \leqslant -c < 0, \tag{5}$$

for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. Similarly to the deterministic situation, condition (5) permits us to introduce a notion of α-hyperbolic times for points in $T^{\mathbb{N}} \times M$ and define a map

$$h_\varepsilon \colon T^{\mathbb{N}} \times M \longrightarrow \mathbb{Z}^+$$

by taking $h_\varepsilon(\underline{t}, x)$ the first α-hyperbolic time for the point $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ (see Section 2). Assuming that h_ε is integrable with respect to $\theta_\varepsilon^{\mathbb{N}} \times m$, then

$$\|h_\varepsilon\|_1 = \sum_{k=0}^{\infty} k\, (\theta_\varepsilon^{\mathbb{N}} \times m)\big(\{(\underline{t}, x)\colon h_\varepsilon(\underline{t}, x) = k\}\big) < \infty. \tag{6}$$

We say that the family $(h_\varepsilon)_{\varepsilon>0}$ has *uniform L^1-tail*, if the series in (6) converges uniformly to $\|h_\varepsilon\|_1$ (as a series of functions of the variable ε).