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THE MINIMAL ENTROPY PROBLEM FOR 3-MANIFOLDS WITH ZERO SIMPLICIAL VOLUME

by

James W. Anderson & Gabriel P. Paternain

Dedicated to Jacob Palis on his sixtieth birthday

Abstract. — In this note, we consider the *minimal entropy problem*, namely the question of whether there exists a smooth metric of minimal (topological) entropy, for certain classes of closed 3-manifolds. Specifically, we prove the following two results.
Theorem A. *Let M be a closed orientable irreducible 3-manifold whose fundamental group contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. The following are equivalent:*
(1) the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;
(2) M admits a geometric structure modelled on $\mathbb{E}^3$ or Nil*;*
(3) M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.
Theorem B. *Let M be a closed orientable geometrizable 3-manifold. The following are equivalent:*
(1) the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;
(2) M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or Nil*;*
(3) M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.

1. Introduction and statement of results

Let M^n be a closed orientable n-dimensional manifold. For a smooth Riemannian metric g on M, let $\mathrm{Vol}(M, g)$ denote the volume of M calculated with respect to g.

Let $\mathrm{h_{top}}(g)$ be the *topological entropy* of the geodesic flow of g, as defined in Section 2.6. Set the *minimal entropy* of M to be

$$\mathrm{h}(M) := \inf\{\mathrm{h_{top}}(g) \mid g \text{ is a smooth metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\}.$$

A smooth metric g_0 with $\mathrm{Vol}(M, g_0) = 1$ is *entropy minimizing* if

$$\mathrm{h_{top}}(g_0) = \mathrm{h}(M).$$

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The *minimal entropy problem* for M is whether or not there exists an entropy minimizing metric on M. Say that the *minimal entropy problem can be solved* for M if there exists an entropy minimizing metric on M. Smooth manifolds are hence naturally divided into two classes: those for which the minimal entropy problem can be solved and those for which it cannot.

There are a number of classes of manifolds for which the minimal entropy problem can be solved. For instance, the minimal entropy problem can always be solved for a closed orientable surface M. For the 2-sphere and the 2-torus, this follows from the fact that both a metric with constant positive curvature and a flat metric have zero topological entropy. For surfaces of higher genus, A. Katok [**11**] proved that each metric of constant negative curvature minimizes topological entropy, and conversely that any metric that minimizes topological entropy has constant negative curvature.

This result of Katok has been generalized to higher dimensions by Besson, Courtois, and Gallot [**1**], as follows. Suppose that M^n ($n \geqslant 3$) admits a locally symmetric metric g_0 of negative curvature, normalized so that $\mathrm{Vol}(M, g_0) = 1$. Then g_0 is the unique entropy minimizing metric up to isometry. Unlike the case of a surface, a locally symmetric metric of negative curvature on a closed orientable n-manifold ($n \geqslant 3$) is unique up to isometry, by the rigidity theorem of Mostow [**18**].

The result of Besson, Courtois, and Gallot [**1**] has itself been generalized by Connell and Farb [**4**] to n-manifolds that admit a complete, finite-volume metric which is locally isometric to a product of negatively curved (rank 1) symmetric spaces of dimension at least 3.

A positive solution to the minimal entropy problem appears to single out manifolds that have either a high degree of symmetry or a low topological complexity. What this means in the context of 3-manifolds will become apparent below. A similar phenomena is observed for closed simply connected manifolds of dimensions 4 and 5: there are essentially only nine manifolds for which the minimal entropy problem can be solved and they can be explicitly listed. These nine manifolds share the property that their loop space homology grows polynomially for any coefficient field, see Paternain and Petean [**21**].

The goal of this note is to classify those closed orientable geometrizable 3-manifolds with zero simplicial volume for which the minimal entropy problem can be solved. Specifically, in Section 4, we prove:

Theorem A. — *Let M be a closed orientable irreducible 3-manifold whose fundamental group contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. The following are equivalent:*

(1) *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*

(2) *M admits a geometric structure modelled on $\mathbb{E}^3$ or* Nil*;*

(3) *M admits a smooth metric g with $\mathrm{h}_{\mathrm{top}}(g) = 0$.*

In Section 5 we prove:

Theorem B. — *Let M be a closed orientable geometrizable 3-manifold. The following are equivalent:*

(1) *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*

(2) *M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or* Nil*;*

(3) *M admits a smooth metric g with* $\mathrm{h_{top}}(g) = 0$.

Recall that the *simplicial volume* of a closed orientable manifold M is defined as the infimum of $\sum_i |r_i|$ where the r_i are the coefficients of a real cycle that represents the fundamental class of M. For 3-manifolds, the positivity of the simplicial volume (which is a homotopy invariant) is closely related to the existence of compact hyperbolizable submanifolds in M. This is described in more detail in Section 2.5.

We close the introduction by describing some of the elements of the proofs of Theorems A and B, and by describing a conjectural picture. We will see in Section 2 that a closed orientable geometrizable 3-manifold M has zero simplicial volume if and only if M has zero minimal entropy. Therefore, the minimal entropy problem can be solved if and only if M admits a smooth metric with zero topological entropy. This in turn forces the fundamental group of M to have subexponential growth. We end up showing that this can occur only if M admits one of the four geometric structures listed in the statement of Theorem B. On the other hand, it is a calculation that the manifolds in the statement of Theorem B carry a metric of zero entropy. The proof of Theorem A follows a similar line, and makes use of the remarkable theorem, due essentially to Thurston, that a manifold satisfying the hypothesis of the theorem is geometrizable. The precise definition of geometrizable manifold is given in Subsection 2.4. Thurston's geometrization conjecture states that every closed orientable 3-manifold is geometrizable.

From this discussion and the above mentioned result of Besson, Courtois and Gallot it seems quite reasonable to speculate that the following statement holds:

Let M be a closed orientable geometrizable 3-manifold. Then, the minimal entropy problem for M can be solved if and only if M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, Nil, *or $\mathbb{H}^3$.*[(1)]

Indeed, suppose that the simplicial volume of M were *not* zero. This would imply that M contains a disjoint collection $H_1, \ldots, H_p$ of compact submanifolds whose interiors each admit a complete hyperbolic structure of finite volume. In particular, it should be that the minimal entropy of M is the maximum of the minimal entropies of the H_k. It

[(1)]Note added in proof: J. Souto (Geometric structures on 3-manifolds and their deformations. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn 2001) has proven this conjecture for all geometrizable prime 3-manifolds would then seem reasonable that an entropy minimizing metric on M would try to be as hyperbolic as possible on the interiors of the H_k and would try as much as possible to be one of the other seven standard 3-dimensional geometries on the components of $M - (H_1 \cup \cdots \cup H_p)$. However, it would seem that the minimizer would have to be singular along the ∂H_k, and so there should be no metric of minimal entropy. Unfortunately, we do not yet know how to make this argument rigorous.

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2. Preliminaries

The purpose of this Section is to present some of the basic material from 3-manifold theory that we will need. We refer the interested reader to Hempel [**8**] for a more detailed introduction to 3-manifold topology. For a more detailed description of Seifert fibered spaces, and of the torus decomposition and the geometrization of 3-manifolds, we also refer the interested reader to the survey articles of Scott [**26**] and Bonahon [**2**], and the references contained therein.

2.1. 3-manifold basics. — We begin with some basic definitions. A 3-manifold is *closed* if it is compact with empty boundary.

An embedded 2-sphere $\mathbb{S}^2$ in a 3-manifold M is *essential* if M does not bound a closed 3-ball in M. A 3-manifold is *irreducible* if it contains no essential 2-sphere.

A 3-manifold is *prime* if it cannot be decomposed as a non-trivial connected sum. That is, M is prime if for every decomposition $M = M_1 \# M_2$ of M as a connected sum, one of M_1 or M_2 is homeomorphic to the standard 3-sphere $\mathbb{S}^3$. Every irreducible 3-manifold is prime, but the converse does not hold. However, the only closed orientable 3-manifold that is prime but not irreducible is $\mathbb{S}^2 \times \mathbb{S}^1$.

We note here that if the closed orientable 3-manifold M contains a non-separating essential 2-sphere, then M can be expressed as the connected sum $M = P \# (\mathbb{S}^2 \times \mathbb{S}^1)$ for some 3-manifold P. Hence, in what follows, we need only consider separating essential 2-spheres in 3-manifolds.

There is an inverse to the operation of connected sum for 3-manifolds, called the *prime decomposition*. The following statement is adapted from Bonahon [**2**], and follows from work of Kneser [**12**] and Milnor [**16**].

Let M be a closed orientable 3-manifold. Then, there exists a compact 2-submanifold Σ of M, unique up to isotopy, so that two conditions hold. First, each component of Σ is an embedded essential separating 2-sphere, and the 2-spheres in Σ are pairwise non-parallel, in that no two 2-spheres in Σ bound an embedded $\mathbb{S}^2 \times [0, 1]$ in M. Second, if $M_0, M_1, \ldots, M_p$ are the closures of the components of $M - \Sigma$, then M_0 is homeomorphic to the 3-sphere $\mathbb{S}^3$ minus finitely many disjoint open 3-balls; while for $k \geqslant 1$, each M_k contains a unique component of Σ, and every separating essential 2-sphere in M_k is parallel to ∂M_k.