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ARTUR AVILA
CARLOS GUSTAVO MOREIRA

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STATISTICAL PROPERTIES OF UNIMODAL MAPS: SMOOTH FAMILIES WITH NEGATIVE SCHWARZIAN DERIVATIVE

by

Artur Avila & Carlos Gustavo Moreira

Abstract. — We prove that there is a residual set of families of smooth or analytic unimodal maps with quadratic critical point and negative Schwarzian derivative such that almost every non-regular parameter is Collet-Eckmann with subexponential recurrence of the critical orbit. Those conditions lead to a detailed and robust statistical description of the dynamics. This proves the Palis conjecture in this setting.

1. Introduction

'The main strategy of the study of all mathematical models is, according to Poincaré, the consideration of each model as a point of a space of different but similar admissible systems' (V. Arnold in [**Ar**]). One of the main concerns of dynamical systems is to establish properties valid for typical systems. Since the space of such systems is usually infinite dimensional, there are of course many concepts of 'typical'. According to [**Ar**] again, 'The most physical genericity notion is defined by Kolmogorov (1954), who suggested to call a property of dynamical systems exceptional, if it holds only on Lebesgue measure zero set of values of the parameters in every (topologically) generic family of systems, depending on sufficiently many parameters'.

In the last decade Palis [**Pa**] described a general program for (dissipative) dynamical systems in any dimension. He conjectured that a typical dynamical system has a finite number of attractors described by physical measures, the union of their basins has full Lebesgue measure, and those physical measures are stochastically stable. Typical was to be interpreted in the Kolmogorov sense: full measure in generic families. Our aim here is to give a proof of this conjecture for an important class of one-dimensional dynamical systems.

Here we consider unimodal maps, that is, continuous maps from an interval to itself which have a unique turning point. More specifically, we consider S-unimodal maps, that is, we assume that the map is C^3 with negative Schwarzian derivative and that the critical point is non-degenerate.

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1.1. The quadratic family. — The basic model for unimodal maps is the quadratic family, $q_a(x) = a - x^2$, where $-1/4 \leqslant a \leqslant 2$ is a parameter. Despite its simple appearance, the dynamics of those maps presents many remarkable phenomena. Restricting to the probabilistic point of view, its richness first became apparent with the work of Jakobson [**J**], where it was shown that a positive measure set of parameters corresponds to quadratic maps with stochastic behavior. More precisely, those parameters possess an absolutely continuous invariant measure (the physical measure) with positive Lyapunov exponent. On the other hand, it was later shown by Lyubich [**L2**] and Graczyk-Swiatek [**GS**] that regular parameters (with a periodic hyperbolic attractor) are (open and) dense. So at least two kinds of very distinct observable behavior are present on the quadratic family, and they alternate in a complicate way.

Besides regular and stochastic behavior, different behavior was shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Those pathologies were shown to be non-observable in [**L3**] and [**MN**]. Finally in [**L4**] it was proved that almost every real quadratic map is either regular or stochastic.

Among stochastic maps, a specific class grabbed lots of attention in the 90's: Collet-Eckmann maps. They are characterized by a positive Lyapunov exponent for the critical value, and gradually they were shown to have 'best possible' near hyperbolic properties: exponential decay of correlations, validity of central limit and large deviations theorems, good spectral properties and zeta functions ([**KN**], [**Y**]). Let us call attention to the robustness of the statistical description, with a good understanding of stochastic perturbations: strong stochastic stability ([**BV**]), rates of convergence to equilibrium ([**BBM**]).

In [**AM1**] the regular or stochastic dichotomy was extended by showing that almost every stochastic map is actually Collet-Eckmann and has polynomial recurrence of its critical point, in particular implying the validity of the above mentioned results.

The position of the quadratic family in the borderline of real and complex dynamics made it a meeting point of many different techniques: most of the deeper results depend on this interaction. It gradually became clear however that studying the quadratic family allows one to obtain results on more general unimodal maps.

1.2. Universality. — Starting with the works of Milnor-Thurston, and also through the discoveries of Feigenbaum and Coullet-Tresser, the quadratic family was shown to be a prototype for other families of unimodal maps which presents universal combinatorial and geometric features. More recently, the result of density of hyperbolicity among unimodal maps was obtained in [**K**] exploiting the validity of this result for quadratic maps.

In [**ALM**], a general method was developed to transfer information from the quadratic family to real analytic families of unimodal maps. It was shown that the decomposition of spaces of analytic unimodal maps according to combinatorial behavior is essentially a codimension-one lamination.

Thinking of two analytic families as transversals to this lamination, one may try to compare the parameter space of both families via the holonomy map. A straightforward application of this method allows one to conclude that the bifurcation pattern of a general analytic family is locally the same as in the quadratic family from the topological point of view (outside of countably many 'bad parameters').

The 'holonomy' method was then successfully applied to extend the regular or stochastic dichotomy from the quadratic family to a general analytic family. The probabilistic point of view presents new difficulties however. First, the statistical properties of two topologically conjugate maps need not correspond by the (generally not absolutely continuous) conjugacy. Fortunately many properties are preserved, in particular the criteria used by Lyubich in his result.

The second difficulty is that the holonomy map is usually not absolutely continuous, so typical combinatorics for the quadratic family may not be typical for other families: it has to be shown that the class of regular or stochastic maps is still typical after application of the holonomy map.

1.3. Results and outline of the proof. — Let us call a k-parameter family good if almost every non-regular parameter is Collet-Eckmann (and satisfies some additional technical conditions). Our goal will be to prove that good families are generic. This question naturally makes sense in different spaces of unimodal maps (corresponding to different degrees of smoothness). We only deal with the last steps of this problem (going from the quadratic family to analytic and then smooth categories), basing ourselves on the building blocks [**L3**], [**L4**], [**ALM**], and [**AM1**].

We start by describing how the holonomy method of [**ALM**] can be applied to generalize the results of [**AM1**] to general analytic families (to put together those two papers we need to do a non-trivial strengthening of [**AM1**]). As a consequence we conclude that essentially all analytic families are good.

To get to the smooth setting (at least C^3, since we are assuming negative Schwarzian derivative), our strategy is different: we show a certain robustness of good families, which together with their denseness (due to the analytic case) will yield genericity. Our main tool is one of the nice properties of Collet-Eckmann maps: persistence of the Collet-Eckmann condition under generic unfolding (a result of [**T1**]). By means of some general argument, we reduce the global result to this local one.

Let us mention that the results of this paper are still valid without the negative Schwarzian derivative assumption (also allowing one to get to C^2 smoothness), see [**A**], [**AM4**]. The techniques are very different however, since we replace the global holonomy method we use here by a local holonomy analysis based on a "macroscopic" version of the infinitesimal perturbation method of [**ALM**]. For analytic maps this also allowed us to obtain better asymptotic estimates which have interesting consequences, for instance pathological measure-theoretical behavior of the lamination by combinatorial classes (see [**AM2**]).

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2. General definitions

2.1. Notation. — Let $I=[-1,1]$ and let B^k be the closed unit ball in $\mathbb{R}^k$ (we will use the notation I for the *dynamical* interval, while B^1 will be reserved for the one-dimensional *parameter* space). We will consider B^k endowed with the Lebesgue measure normalized so that $|B^k|=1$. Let $C^r(I)$ denote the space of C^r maps $f: I\to\mathbb{R}$.

By a *unimodal map* we will mean a smooth (at least C^2) symmetric (even) map $f: I\to I$ with a unique critical point at 0 such that $f(-1)=-1$, $Df(-1)\geqslant 1$, and if $Df(-1)=1$ then $D^2f(-1)<0$. If f is C^3, we define the Schwarzian derivative on $I\smallsetminus\{0\}$ as

$$Sf=\frac{D^3f}{Df}-\frac{3}{2}\left(\frac{D^2f}{Df}\right)^2.$$

For $a>0$, let $\Omega_a\subset\mathbb{C}$ denote an a neighborhood I.

Let $\mathcal{A}_a$ denote the space of holomorphic maps on Ω_a which have a continuous extension to $\partial\Omega_a$, satisfying $\phi(z)=\phi(-z)$, $\phi(-1)=\phi(1)=-1$ and $\phi'(0)=0$.

Notice that $\mathcal{A}_a$ is a closed affine subspace of the Banach space of bounded holomorphic maps of Ω_a. We endow it with the induced metric and affine structure.

We define $\mathcal{A}_a^{\mathbb{R}}\subset\mathcal{A}_a$ the space of maps which are real symmetric.

2.2. More on unimodal maps. — A C^3 unimodal map such that $Sf<0$ on $I\smallsetminus\{0\}$ and such that its critical point is non-degenerate (that is, $D^2f\neq 0$) will be called a *S-unimodal map.*

We say that x is a periodic orbit (of period n) for f if $f^n(x)=x$ and $n\geqslant 1$ is minimal with this property. In this case we define $Df^n(x)$ as the multiplier of x. Notice that this definition depends only on the orbit of x. We say that x is hyperbolic if $|Df^n(x)|\neq 1$.

A unimodal map is called *regular* (or hyperbolic) if all periodic orbits are hyperbolic and the iterates of the critical point converge to an attracting periodic orbit. This condition is C^2-open, moreover a S-unimodal map is regular if and only if it has a hyperbolic periodic attractor (see [**MvS**]).

A *k-parameter family* of unimodal maps is a map $F: B^k\times I\to I$ such that for $p\in B^k$, $f_p(x)=F(p,x)$ is a unimodal map. Such a family is said to be C^n or analytic, according to F being C^n or analytic. We introduce the natural topology in spaces of smooth families (C^n with $n=2,\dots,\infty$), but do not introduce any topology in the