Astérisque

ARTUR O. LOPES
PHILIPPE THIEULLEN

Sub-actions for Anosov diffeomorphisms

Astérisque, tome 287 (2003), p. 135-146
<http://www.numdam.org/item?id=AST_2003__287__135_0>

Astérisque
287, 2003, p. 135–146

SUB-ACTIONS FOR ANOSOV DIFFEOMORPHISMS

by

Artur O. Lopes & Philippe Thieullen

Dedicated to Jacob Palis

Abstract. — We show a positive Livsic type theorem for $\mathcal{C}^2$ Anosov diffeomorphisms f on a compact boundaryless manifold M and Hölder observables A. Given $A : M \to \mathbb{R}$, α-Hölder, we show there exist $V : M \to \mathbb{R}$, β-Hölder, $\beta < \alpha$, and a probability measure μ, f-invariant such that

$$A \leqslant V \circ f - V + \int A \, d\mu.$$

We apply this inequality to prove the existence of an open set $\mathcal{G}_\beta$ of β-Hölder functions, β small, which admit a unique maximizing measure supported on a periodic orbit. Moreover the closure of $\mathcal{G}_\beta$, in the β-Hölder topology, contains all α-Hölder functions, α close to one.

1. Introduction

We consider a compact riemannian manifold M of dimension $d \geqslant 2$ without boundary and a $\mathcal{C}^2$ transitive Anosov diffeomorphism $f : M \to M$. The tangent bundle TM admits a continuous Tf-invariant splitting $TM = E^u \bigoplus E^s$ of expanding and contracting tangent vectors. We assume M is equiped with a riemannian metric and there exists a constant $C(M)$, depending only on M and the metric and constants depending on f

$$0 < \Lambda_s < \lambda_s < 1 < \lambda_u < \Lambda_u$$

such that for all $n \in \mathbb{Z}$

$$\begin{cases} C(M)^{-1}\lambda_u^n \leqslant \|T_x f^n \cdot v\| \leqslant C(M)\Lambda_u^n & \text{for all } v \text{ in } E_x^u, \\ C(M)^{-1}\Lambda_s^n \leqslant \|T_x f^n \cdot v\| \leqslant C(M)\lambda_s^n & \text{for all } v \text{ in } E_x^s. \end{cases}$$

2000 ***Mathematics Subject Classification.*** — 37D20.
Key words and phrases. — Anosov diffeomorphisms, minimizing measures.

A.L.: Partially supported by PRONEX-CNPq - Sistemas Dinâmicas and Institute of Millenium - IMPA.
Ph.T.: Partially supported by CNRS URA 1169.

Livsic theorem [**5**] asserts that, if $A : M \to M$ is a given Hölder function and satisfies $\int A\,d\mu = 0$ for all f-invariant probability measure μ, then A is equal to a coboundary V (which is Hölder too), that is:

$$A = V \circ f - V.$$

What hapens if we only assume $\int A\,d\mu \geqslant 0$ for all f-invariant probability measure μ? We denote by $\mathcal{M}(f)$, the set of f-invariant probability measures and $m(A,f) = \sup\big\{ \int A\,d\mu \mid \mu \in \mathcal{M}(f)\big\}$.

For a β-Holder function V

$$\text{Höld}_\beta(V) = \sup_{0<d(x,y)} \Big\{ \frac{|V(x)-V(y)|}{d(x,y)^\beta} \Big\}.$$

We prove the following:

Theorem 1. — *Let $f : M \to M$ be a $\mathcal{C}^2$ transitive Anosov diffeomorphism on a compact manifold M without boundary. For any given α-Hölder function $A : M \to \mathbb{R}$, there exists a β-Hölder function $V : M \to \mathbb{R}$, that we call sub-action, such that:*

$$A \leqslant V \circ f - V + m(A,f),$$

and

$$\beta = \alpha \frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}, \quad \text{Höld}_\beta(V) \leqslant \frac{C(M)}{\min(1-\lambda_u^{-\alpha}, 1-\lambda_s^{\alpha})^2}\,\text{Höld}_\alpha(A)$$

where $C(M)$ is some constant depending only on M and the metric.

By analogy with Hamiltonian mechanics and the way we define V from A, we may interpret A as a lagrangian and V as a sub-action. This result extends a similar one we obtained in [**4**] for expanding maps of the circle (see [**2**] [**6**] for related results). The same techniques of [**4**] also apply for the one-directional shift as it is mentioned in [**4**].

The proof we give here is for bijective smooth systems, and we obtain V continuous in all M. Our result can not be derived (via Markov partition) directly from an analogous result for the bi-directional shift.

Corollary 2. — *The hypothesis are the same as in theorem 1. The following statements are equivalent:*

(i) $A \geqslant V \circ f - V$ *for some bounded measurable function* V,
(ii) $\int A\,d\mu \geqslant 0$ *for all f-invariant probability measure* μ,
(iii) $\sum_{k=0}^{p-1} A \circ f^k(x) \geqslant 0$ *for all $p \geqslant 1$ and point x periodic of period* p,
(iv) $A \geqslant V \circ f - V$ *for some Hölder function* V.

The proof of that corollary is straightforward and uses (for (iii) $\Rightarrow$ (ii)) the fact that the convex hull of periodic measures is dense in the set of all f-invariant probability measures for topological dynamical systems satisfying the shadowing lemma (see Lemma 5). F. Labourie suggested to us the following corollary:

Corollary 3. — *The hypothesis are the same as in theorem 1. If A satisfies $\int A\,d\mu \geqslant 0$ for all $\mu \in \mathcal{M}(f)$ and $\sum_{k=0}^{p-1} A \circ f^k(x) > 0$ for at least one periodic orbit x of period p then $\int A\,d\lambda > 0$ for all probability measure λ giving positive mass to any open set.*

Again the proof is straitforward: $R = A - V \circ f + V \geqslant 0$ for some continuous V and $\int R\,d\lambda = 0$ for such a measure λ implies $R = 0$ everywhere and in particular $\sum_{k=0}^{p-1} A \circ f(x) = 0$ for all periodic orbit x.

Any measure μ satisfying $\int A\,d\mu = m(A, f)$ is called a maximizing measure and since A is continuous, such a measure always exists. It is then natural to ask the following two questions: For which A, the set of maximizing measures is reduced to a single measure ? In the case there exists a unique maximizing measure, to what kind of compact set, the support of this measure looks like ?

The following theorem gives a partial answer for "generic" functions A.

Theorem 4. — *Let $f : M \to M$ be a $\mathcal{C}^2$ transitive Anosov diffeomorphism and $\beta < \ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$. Then there exists an open set $\mathcal{G}_\beta$ of β-Hölder functions (open in the $\mathcal{C}^\beta$-topology) such that:*

(i) *any A in $\mathcal{G}_\beta$ admits a unique maximizing measure μ_A;*

(ii) *the support of μ_A is equal to a periodic orbit and is locally constant with respect to $A \in \mathcal{G}_\beta$;*

(iii) *any α-Hölder function with $\alpha > \beta \ln(\Lambda_u/\lambda_s)/\ln(1/\lambda_s)$ is contained in the closure of $\mathcal{G}_\beta$ (the closure is taken with respect to the $\mathcal{C}^\beta$-topology).*

The proof of Theorem 4 is a simplification of what we gave in [**4**] in the one-dimensional setting. The existence of sub-actions is in both cases the main ingredient of the proof.

Now we will concentrate in one of our main results, namely, Theorem 1; the basic idea is the following: given a finite covering of M by open sets $\{U_1, \dots, U_l\}$ with sufficiently small diameter, we construct a Markov covering (and not a Markov partition) $\{R_1, \dots, R_l\}$ of rectangles: each R_i contains U_i and satisfies

$$x \in U_i \cap f^{-1}(U_j) \Longrightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j),$$

where $W^s(x, R_i)$ denotes the local stable leaf through x restricted to R_i. We then associate to each R_i a local sub-action V_i, defined on R_i by:

$$V_i(x) = \sup\big\{S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geqslant 0,\ y \in W^s(x, R_i)\big\}$$

where $\Delta^s(y, x)$ is a kind of cocycle along the stable leaf $W^s(x)$:

$$\Delta^s(y, x) = \sum_{n \geqslant 0} (A \circ f^n(y) - A \circ f^n(x)),$$

and where $S_n(A - m) = \sum_{k=0}^{n-1} (A - m) \circ f^k$.

This family $\{V_1, \dots, V_l\}$ of local sub-actions satisfies the inequality:

$$x \in U_i \cap f^{-1}(U_j) \Longrightarrow V_i(x) + A(x) - m \leqslant V_j \circ f(x)$$

and enable us to construct a global sub-action V:

$$V(x) = \sum_{i=1}^{l} \theta_i(x) V_i(x)$$

where $\{\theta_1, \ldots, \theta_l\}$ is a smooth partition of unity associated to the covering $\{U_1, \ldots, U_l\}$. The main difficulty is to prove that each V_i is Hölder on R_i.

2. Existence of sub-actions

We continue our description of the dynamics of transitive Anosov diffeomorphisms (for details information, see Bowen's monography [**3**]). All the results we are going to use depend on a small constant of expansiveness $\varepsilon^* > 0$ (by definition this constant says that any pseudo-orbit can be followed by true orbits) depending on f and M in the following way:

$$\varepsilon^* = C(M)^{-1} \min\Big(\frac{\lambda_u - 1}{\|D^2 f\|_\infty}, \frac{1 - \lambda_s}{\|D^2 f\|_\infty}\Big)$$

where $C(M) \geqslant 1$ is a constant depending only on M and the riemannian metric. At each point x, one can define its local stable manifold $W^s_\varepsilon(x)$ for every $\varepsilon < \varepsilon^*$:

$$W^s_\varepsilon(x) = \big\{y \in M \mid d(f^n(x), f^n(y)) \leqslant \varepsilon \;\forall\, n \geqslant 0\big\}$$

which are $\mathcal{C}^2$ embeded closed disks of dimension $d^s = \dim E^s_x$ and tangent to E^s_x. In the same manner, $W^u_\varepsilon(x)$ is defined replacing f by f^{-1}. If two points x, y are close enough, $d(x, y) < \delta$, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ have a unique point in common, called $[x, y]$:

$$[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y) = W^s_{\varepsilon^*}(x) \cap W^u_{\varepsilon^*}(y),$$

where $\varepsilon = K^*\delta$ and K^* is again a large constant depending on M and f:

$$K^* = \frac{C(M)}{\min\,(1 - \lambda_u^{-1}, 1 - \lambda_s)}.$$

This estimate is in fact a particular case of Bowen's shadowing lemma:

***Lemma 5* (Bowen).** — *If δ is small enough, $\delta < \varepsilon^*/K^*$, if $(x_n)_{n\in\mathbb{Z}}$ is a bi-infinite δ-pseudo-orbit, that is, $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$, then there exists a unique true orbit $\{f^n(x)\}_{n\in\mathbb{Z}}$ which ε-shadow $(x_n)_{n\in\mathbb{Z}}$, that is $d(f^n(x), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$ with $\varepsilon = K^*\delta$.*

This lemma (see [**3**]) for proof) is the main ingredient for constructing (dynamical) rectangles. A rectangle R is a closed set of diameter less than ε^*/K^* satisfying:

$$x, y \in R \Longrightarrow [x, y] \in R.$$

We will not use the notion of proper rectangles but will use instead the notion of Markov covering.