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ON SOME APPROXIMATIONS OF THE QUASI-GEOSTROPHIC EQUATION

by

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Abstract. — For two-dimensional quasi-geostrophic equation in Fourier space, we propose a new type approximation representing itself some quasi-linear equation. Natural finite dimensional approximations of this equation are investigated in the article.

1. Introduction

The main difficulty in the proof of existence and uniqueness of solutions of hydrodynamical equations is the lack of understanding of the role played by non-linear, or Eulerian, terms. In Fourier space these terms describe the expansion of initial excitations of Fourier modes but the way how this process goes is in general unclear.

In this paper we propose an approach which leads to some simplifications of the original equations with the belief that the processes of expansion remain the same. Our equations have natural finite-dimensional approximations which are systems of ODE and are easier to tackle.

We restrict ourselves to the two-dimensional quasi-geostrophic equation (QGE) for an unknown function $u(k,t), k = (k_1, k_2) \in R^2$ which in Fourier space has the form (see [**1**], [**2**])

$$\frac{\partial u(k,t)}{\partial t} = \int_{R^2} \frac{((k')^{\perp}, k-k')}{|k-k'|} u(k',t)u(k-k',t)dk' - \nu |k|^{2\alpha} u(k,t) \tag{1}$$

Here $|k| = (k_1^2 + k_2^2)^{1/2}$, $k^{\perp} = (-k_2, k_1)$, $\nu \geqslant 0$ is the viscosity and we are interested in even solutions $u(-k,t) = u(k,t)$. It is well-known that the mathematical difficulties

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related to (1) are in many respects similar to the well-known difficulties for the $3D$-Navier-Stokes system.

The main case is $\alpha = 1$. For $0 < \alpha < 1$ we obtain the so-called generalized QGE, which we also consider in this paper.

We are interested in solutions which are smooth in k and decay at infinity rather slowly. Our main assumption says that for such solutions the main contribution to the integral in (1) comes from $|k'| \ll |k|$ or $|k - k'| \ll |k|$. For $|k'| \ll |k|$ we can write

$$\frac{((k')^{\perp}, k - k')}{|k - k'|} = ((k')^{\perp}, k) \left(\frac{1}{|k|} - (\nabla \frac{1}{|k|}, k') \right) + \cdots$$

where dots mean terms of a smaller order of magnitude. Thus for $|k'| \ll |k|$ we keep the term

$$\int_{R^2} ((k')^{\perp}, k) \left(\frac{1}{|k|} - (\nabla \frac{1}{|k|}, k') \right) u(k', t)(u(k, t) - (\nabla u(k, t), k'))dk'$$
$$= \int_{R^2} \frac{((k')^{\perp}, k)}{|k|} u(k', t)u(k, t)dk' - u(k, t) \int_{R^2} ((k')^{\perp}, k)(\nabla \frac{1}{|k|}, k')u(k', t)dk'$$
$$- \int_{R^2} \frac{((k')^{\perp}, k)}{|k|} (\nabla u(k, t), k')u(k', t)dk' + \int_{R^2} ((k')^{\perp}, k)(\nabla \frac{1}{|k|}, k')(\nabla u(k, t), k'))u(k', t)dk'$$

The first and the last integrals are zero because the integrands are odd functions of k'. For $|k - k'| \ll k$ put $k'' = k - k'$. Then

$$\int_{R^2} \frac{((k - k'')^{\perp}, k)}{|k''|} u(k - k'', t)u(k'', t)dk''$$
$$= \int_{R^2} \frac{((k^{\perp}), k'')}{|k''|} (u(k, t) - (\nabla u(k, t), k''))u(k'', t)dk'' + \cdots$$
$$= \int_{R^2} \frac{((k^{\perp}), k'')}{|k''|} u(k, t)u(k'', t)dk'' - \int_{R^2} \frac{((k^{\perp}), k'')}{|k''|} (\nabla u(k, t), k'')u(k'', t)dk'' + \cdots$$

Again dots mean terms of a smaller order of magnitude. The first integral is zero by the same reasons as above, i.e. the parity of the integrand. Thus our approximating equation takes the form

$$(2) \quad \frac{\partial u(k, t)}{\partial t} = -u(k, t) \int_{R^2} ((k')^{\perp}, k)(\nabla \frac{1}{|k|}, k')u(k', t)dk'$$
$$- \int_{R^2} \frac{((k')^{\perp}, k)}{|k|} u(k', t)\,(k', \nabla u(k, t))\,dk'$$
$$- \int_{R^2} \frac{((k)^{\perp}, k')}{|k'|} u(k', t)\,(k', \nabla u(k, t))\,dk' - \nu|k|^{2\alpha} u(k, t)$$

The equation (2) does not satisfy the energy estimate but apparently remains dissipative because of viscosity. Let us rewrite (2) as follows:

$$(3)\quad \frac{\partial u(k,t)}{\partial t} = -u(k,t)\int_{R^2}((k')^{\perp},k)(\nabla\frac{1}{|k|},k')u(k',t)dk' \\ - \int_{R^2}((k')^{\perp},k)\left[\frac{1}{|k|}+\frac{1}{|k'|}\right]u(k',t)\,(k',\nabla u(k,t))\,dk' - \nu|k|^{2\alpha}u(k,t)$$

The equation (3) is a first order quasi-linear equation whose coefficients are global functions of u. Take the first term in (3):

$$I_1(t) = \int_{R^2}((k')^{\perp},k)\left(\nabla\frac{1}{|k|},k'\right)u(k',t)dk'$$

We have

$$(k')^{\perp} = (-k_2',k_1');\nabla\frac{1}{|k|} = \left(-\frac{k_1}{(k_1^2+k_2^2)^{3/2}},-\frac{k_2}{(k_1^2+k_2^2)^{3/2}}\right).$$

Therefore

$$\begin{aligned} I_1(t) &= -\int(k_1'k_2-k_2'k_1)\frac{k_1k_1'+k_2k_2'}{|k|^3}\,u(k':t)\,dk' \\ &= -\frac{k_2^2}{|k|^3}\int k_1'k_2'u(k',t)dk' + \frac{k_1^2}{|k|^3}\int k_1'k_2'u(k',t)dk' \\ &\quad \frac{k_1k_2}{|k|^3}\int((k_1')^2-(k_2')^2)u(k',t)dk'. \end{aligned}$$

Denote

$$a_1(t) = \int k_1^2u(k,t)dk;\quad a_2(t) = \int k_2^2u(k,t)dk;\quad a_3(t) = \int k_1k_2u(k,t)dk;$$

Then

$$I_1(t) = \frac{k_1^2-k_2^2}{|k|^3}a_3 - \frac{k_1k_2}{|k|^3}(a_1-a_2)$$

Consider

$$I_2(t) = \int((k')^{\perp},k)\frac{1}{|k|}u(k',t)(k',\nabla u(k,t))dk'$$

We have

$$\begin{aligned} I_2(t) &= \int dk'\frac{k_2k_1'-k_1k_2'}{|k|}u(k',t)(k_1'\frac{\partial u(k,t)}{\partial k_1}+k_2'\frac{\partial u(k,t)}{\partial k_2}) \\ &= \frac{\partial u(k,t)}{\partial k_1}\left[-\frac{k_1}{|k|}\int k_1'k_2'u(k',t)dk' + \frac{k_2}{|k|}\int(k_1')^2u(k',t)dk'\right] \\ &\quad + \frac{\partial u(k,t)}{\partial k_2}\left[-\frac{k_1}{|k|}\int(k_2')^2u(k',t)dk' + \frac{k_2}{|k|}\int k_1'k_2'u(k',t)dk'\right] \\ &= \frac{\partial u(k,t)}{\partial k_1}\left[-\frac{k_1}{|k|}a_3+\frac{k_2}{|k|}a_1\right] + \frac{\partial u(k,t)}{\partial k_2}\left[-\frac{k_1}{|k|}a_2+\frac{k_2}{|k|}a_3\right] \end{aligned}$$

The last term

$$\begin{aligned}I_3(t) &= \int \frac{((k')^{\perp}, k)}{|k'|} u(k', t) \left(k_1' \frac{\partial u(k,t)}{\partial k_1} + k_2' \frac{\partial u(k,t)}{\partial k_2} \right) dk' \\ &= \frac{\partial u(k,t)}{\partial k_1} \left(-k_1 \int \frac{k_1' k_2'}{|k'|} u(k',t) dk' + k_2 \int \frac{(k_1')^2}{|k'|} u(k',t) dk' \right) \\ &\quad + \frac{\partial u(k,t)}{\partial k_2} \left(-k_1 \int \frac{(k_2')^2}{|k'|} u(k',t) dk' + k_2 \int \frac{k_1' k_2'}{|k'|} u(k',t) dk' \right)\end{aligned}$$

Denote

$$b_1(t) = \int \frac{(k_1')^2}{|k'|} u(k',t) dk', \quad b_2(t) = \int \frac{(k_2')^2}{|k'|} u(k',t) dk', \quad b_3(t) = \int \frac{k_1' k_2'}{|k'|} u(k',t) dk',$$

$$\begin{aligned}h_0(k,t) &= \left[\frac{k_2^2 - k_1^2}{|k|^3} a_3 + \frac{k_1 k_2}{|k|^3} (a_1 - a_2) - \nu |k|^{2\alpha} \right], \\ h_1(k,t) &= -\left[\frac{k_1}{|k|} a_3 - \frac{k_2}{|k|} a_1 + k_1 b_3 - k_2 b_1 \right], \\ h_2(k,t) &= -\left[\frac{k_1}{|k|} a_2 - \frac{k_2}{|k|} a_3 + k_1 b_2 - k_2 b_3 \right].\end{aligned}$$

Then the equation (3) takes its final form:

$$\frac{\partial u(k,t)}{\partial t} + \frac{\partial u(k,t)}{\partial k_1} h_1(k,t) + \frac{\partial u(k,t)}{\partial k_2} h_2(k,t) = h_0(k,t) u$$

or

$$\frac{du(k,t)}{dt} = h_0(k,t) u(k,t) \tag{4}$$

where

$$\begin{aligned}\frac{dk_1}{dt} &= h_1(k,t) \\ \frac{dk_2}{dt} &= h_2(k,t).\end{aligned} \tag{5}$$

However, we should not forget that the coefficients a_i, b_i are also functions of unknowns k and u. (5) are the equations for characteristics of our quasi-linear equation. We can think about them as curves along which the non-linearity spreads. Denote by S^{t_1, t_2} the family of shifts along solutions of (5). Then

$$u(k, t_2) = u(k(t_1), t_1) \exp\left(\int_{t_1}^{t_2} h_0(k(\tau), \tau)) d\tau \right), \tag{6}$$

where $k(\tau) = S^{t_1, \tau} k(t_1)$.

Corollary 1. *The sign of u is preserved along the characteristics of* (5).

Proof. — Follows immediately from (6). □