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ON RANDOM AND MEAN EXPONENTS FOR UNITARILY INVARIANT PROBABILITY MEASURES ON $\mathbb{GL}_n(\mathbb{C})$

by

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Dedicated to Jacob Palis for his sixtieth birthday.

Abstract. — We consider unitarily invariant probability measures on $\mathbb{GL}_n(\mathbb{C})$ and compare the mean of the logs of the moduli of the eigenvalues of the matrices to the Lyapunov exponents of random matrix products independently drawn with respect to the measure. We prove that the former is always greater or equal to the latter.

1. Introduction

Given a probability measure μ on the space of invertible $n \times n$ complex matrices satisfying a mild integrability condition, we have, by Oseledec's Theorem, n random exponents $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n \geqslant -\infty$ such that for almost every sequence $\ldots g_k \ldots g_1 \in \mathbb{GL}_n(\mathbb{C})$ the limit $\lim \frac{1}{k} \log \|g_k \ldots g_1 v\|$ exists for every $v \in \mathbb{C}^n \setminus \{0\}$ and equals one of the r_i, $i = 1 \ldots n$, see Gol'dsheid and Margulis [**4**] or Ruelle [**8**] or Oseledec [**7**]. The numbers $r_1, \ldots, r_n$ are called Lyapunov exponents. In our context we may call them random Lyapunov exponents or even just random exponents. If the measure is concentrated on a point A, these numbers $\lim \frac{1}{n} \log \|A^n v\|$ are $\log |\lambda_1|, \ldots, \log |\lambda_n|$ where $\lambda_i(A) = \lambda_i$, $i = 1 \ldots n$, are the eigenvalues of A written with multiplicity and $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$.

The integrability condition for Oseledec's Theorem is

$$g \in \mathbb{GL}_n(\mathbb{C}) \to \log^+(\|g\|) \text{ is } \mu - \text{integrable}$$

where for a real valued function f, $f^+ = \max[0, f]$. Here we will assume more so that all our integrals are defined and finite, namely:

$$(*) \qquad g \in \mathbb{GL}_n(\mathbb{C}) \to \log^+(\|g\|) \text{ and } \log^+(\|g^{-1}\|) \text{ are } \mu\text{-integrable.}$$

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We will prove:

Theorem 1. — *If μ is a unitarily invariant measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then, for $k = 1, \dots, n$,*

$$\int_{A\in\mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{k} \log|\lambda_i(A)| d\mu(A) \geqslant \sum_{i=1}^{k} r_i.$$

By unitary invariance we mean $\mu(U(X)) = \mu(X)$ for all unitary transformations $U \in \mathbb{U}_n(\mathbb{C})$ and all μ-measurable $X \subseteq \mathbb{GL}_n(\mathbb{C})$.

Corollary 2

$$\int_{A\in\mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{n} \log^+|\lambda_i(A)| d\mu(A) \geqslant \sum_{i=1}^{n} {r_i}^+.$$

Theorem 1 is not true for general measures on $\mathbb{GL}_n(\mathbb{C})$ or $\mathbb{GL}_n(\mathbb{R})$ even for $n = 2$. Consider

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and give probability $1/2$ to each. Then the left hand integral is zero but as is easily seen the right hand sum is positive. So, in this case the inequality goes the other way. We do not know a characterization of measures which make Theorem 1 valid. We would find such a characterization interesting.

The numbers $\sum_{i=1}^{k} r_i$ have a direct geometric interpretation. Let $\mathbb{G}_{n,k}(\mathbb{C})$ denote the Grassmannian manifold of k dimensional vector subspaces in $\mathbb{C}^n$, $A|G_{n,k}$ the restriction of A to the subspace $G_{n,k}$ and ν the natural unitarily invariant probability measure on $\mathbb{G}_{n,k}(\mathbb{C})$.

Theorem 3. — *If μ is a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then,*

$$\sum_{i=1}^{k} r_i = \int_{A\in\mathbb{GL}_n(\mathbb{C})} \int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})} \log|\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}) d\mu(A).$$

We may then restate Theorem 1 in the form we prove it.

Theorem 4. — *If μ is a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then, for $k = 1, \dots, n$*

$$\int_{A\in\mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{k} \log|\lambda_i(A)| d\mu(A) \geqslant \int_{A\in\mathbb{GL}_n(\mathbb{C})} \int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})} \log|\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}) d\mu(A).$$

There is a considerable literature on random Lyapunov exponents and quite general criteria which guarantee that they are non-zero and even distinct. According to Bougerol and Lacroix in 1985 in [**2**] "The subject matter initiated by Bellman was fully developed by Furstenberg, Guivarc'h, Kesten, Le Page and Raugi." We refer to [**2**] for references prior to 1985 and to three others: Gol'dsheid and Margulis [**4**], Guivarc'h and Raugi [**5**] and Ledrappier [**6**].

Our interest in Theorem 1 and Theorem 4 was motivated by some questions in dynamical systems theory, see Burns, Pugh, Shub and Wilkinson [**3**]. Theorem 1 for $k = 1$, the orthogonal group and $\mathbb{GL}_n(\mathbb{R})$ was raised there.

We also get a version of Theorem 4 without the logarithms.

Theorem 5. — *Let μ be a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ and $1 \leqslant k \leqslant n$. Then*

$$\int_{A\in\mathbb{GL}_n(\mathbb{C})} \prod_{i=1}^{k} |\lambda_i(A)| d\mu(A) \geqslant \int_{A\in\mathbb{GL}_n(\mathbb{C})} \int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})} |\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}) d\mu(A).$$

There is a special case of Theorems 4 and 5 that is good to keep in mind. Our proof relies it.

Let $A \in \mathbb{GL}_n(\mathbb{C})$ and μ be the Haar measure on $\mathbb{U}_n(\mathbb{C})$ (the unitary subgroup of $\mathbb{GL}_n(\mathbb{C})$) normalized to be a probability measure. In this case Theorem 5 becomes:

Theorem 6. — *Let $A \in \mathbb{GL}_n(\mathbb{C})$. Then, for $1 \leqslant k \leqslant n$,*

$$\int_{U\in\mathbb{U}_n(\mathbb{C})} \sum_{i=1}^{k} \log|\lambda_i(UA)| d\mu(U) \geqslant \int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})} \log|\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k})$$

and

$$\int_{U\in\mathbb{U}_n(\mathbb{C})} \prod_{i=1}^{k} |\lambda_i(UA)| d\mu(U) \geqslant \int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})} |\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}).$$

When $k = 1$, $|\lambda_1(UA)| = \rho(UA)$ is the spectral radius of UA. The Grassmannian manifold is identical to the complex projective space $\mathbb{P}_{n-1}(\mathbb{C})$. Integration on this manifold can be reduced to the unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{R}^{2n}$ so that

Corollary 7. — *Let $A \in \mathbb{GL}_n(\mathbb{C})$. Then*

$$\int_{U\in\mathbb{U}_n(\mathbb{C})} \log|\rho(UA)| d\mu(U) \geqslant \int_{x\in\mathbb{S}^{2n-1}} \log\|Ax\| d\nu(x)$$

and

$$\int_{U\in\mathbb{U}_n(\mathbb{C})} |\rho(UA)| d\mu(U) \geqslant \int_{x\in\mathbb{S}^{2n-1}} \|Ax\| d\nu(x).$$

We expect a similar result for orthogonally invariant probability measures on $\mathbb{GL}_n(\mathbb{R})$ but we have not proven it. Here we content ourselves with the case $n = 2$.

Theorem 8. — *Let μ be a probability measure on $\mathbb{GL}_2(\mathbb{R})$ satisfying*

$$g \in \mathbb{GL}_2(\mathbb{R}) \to \log^+(\|g\|) \text{ and } \log^+(\|g^{-1}\|) \text{ are } \mu\text{-integrable.}$$

(a) *If μ is a $\mathbb{SO}_2(\mathbb{R})$ invariant measure on $\mathbb{GL}_2^+(\mathbb{R})$ then,*

$$\int_{A\in\mathbb{GL}_2^+(\mathbb{R})} \log|\lambda_1(A)|d\mu(A) = \int_{A\in\mathbb{GL}_2^+(\mathbb{R})}\int_{x\in\mathbb{S}^1} \log\|Ax\|d\mathbb{S}^1(x)d\mu(A).$$

(b) *If μ is a $\mathbb{SO}_2(\mathbb{R})$ invariant measure on $\mathbb{GL}_2^-(\mathbb{R})$, whose support is not contained in $\mathbb{RO}_2(\mathbb{R})$ i.e. in the set of scalar multiples of orthogonal matrices, then*

$$\int_{A\in\mathbb{GL}_2^-(\mathbb{R})} \log|\lambda_1(A)|d\mu(A) > \int_{A\in\mathbb{GL}_2^-(\mathbb{R})}\int_{x\in\mathbb{S}^1} \log\|Ax\|d\mathbb{S}^1(x)d\mu(A).$$

Here $\mathbb{GL}_2^+(\mathbb{R})$ (resp. $\mathbb{GL}_2^-(\mathbb{R})$) is the set of invertible matrices with positive (resp. negative) determinant. Theorem 8 is proved in section 5.

2. A More General Theorem

Theorem 4 is actually a special case of the much more general Theorem 11 below. Before we state Theorem 11 we need some preliminaries.

A flag F in $\mathbb{C}^n$ is a sequence of vector subspaces of $\mathbb{C}^n$: $F = (F_1, F_2, \ldots, F_n)$, with $F_i \subset F_{i+1}$ and Dim $F_i = i$. The space of flags is called the flag manifold and we denote it by $\mathbb{F}_n(\mathbb{C})$. Now it is easy to see that $\mathbb{F}_n(\mathbb{C})$ may be represented by $\mathbb{GL}_n(\mathbb{C})/\mathbb{R}_n(\mathbb{C})$ or by $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$, where $\mathbb{R}_n(\mathbb{C})$ is the subgroup of $\mathbb{GL}_n(\mathbb{C})$ of upper triangular matrices and $\mathbb{T}^n(\mathbb{C})$ is the subgroup of $\mathbb{GL}_n(\mathbb{C})$ consisting of diagonal matrices with complex numbers of modulus 1, so $\mathbb{T}^n(\mathbb{C}) = \mathbb{U}_n(\mathbb{C}) \cap \mathbb{R}_n(\mathbb{C})$. Regarding $\mathbb{F}_n(\mathbb{C})$ as $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$ we see that $\mathbb{F}_n(\mathbb{C})$ has a natural $\mathbb{U}_n(\mathbb{C})$-invariant probability measure.

An invertible linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ naturally induces a map $A_\sharp$ on flags by

$$A_\sharp(F_1, F_2, \ldots, F_n) = (AF_1, AF_2, \ldots, AF_n).$$

The flag manifold and the action of a linear map A on $\mathbb{F}_n(\mathbb{C})$ is closely related to the QR algorithm, see Shub and Vasquez [**9**] for a discussion of this. In particular if F is a fixed flag for A i.e. $A_\sharp F = F$, then A is upper triangular in a basis corresponding to the flag F, with the eigenvalues of A appearing on the diagonal in some order: $\lambda_1(A, F), \ldots, \lambda_n(A, F)$.

Let

$$\mathbb{G} = \{A \in \mathbb{GL}_n(\mathbb{C}) \ : \ |\lambda_1(A)| > |\lambda_2(A)| > \cdots > |\lambda_n(A)|\}.$$

Then, there is a unique flag F such that $A_\sharp(F) = F$ and such that $\lambda_i(A, F) = \lambda_i(A)$ for $i = 1, \ldots, n$. We call this flag the QR flag of A and let $QR : \mathbb{G} \to \mathbb{F}_n(\mathbb{C})$ be the map which associates to $A \in \mathbb{G}$ its QR flag. It follows from Shub-Vasquez [**9**] and the discussion of fixed point manifolds below that QR is a smooth mapping.

Now fix $A \in \mathbb{GL}_n(\mathbb{C})$, define $\mathbb{U}_n(\mathbb{C})A = \{UA \ : \ U \in \mathbb{U}_n(\mathbb{C})\}$ and consider $\mathbb{G}_A = \mathbb{G} \cap (\mathbb{U}_n(\mathbb{C})A)$. Assume that $\mathbb{G}_A \neq \varnothing$. If we restrict QR to $\mathbb{G}_A$ then $QR : \mathbb{G}_A \to \mathbb{F}_n(\mathbb{C})$