Astérisque

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Astérisque, tome 287 (2003), p. 231-249

<http://www.numdam.org/item?id=AST_2003__287__231_0>

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287, 2003, p. 231–249

ON THE DIVERGENCE OF GEODESIC RAYS IN MANIFOLDS WITHOUT CONJUGATE POINTS, DYNAMICS OF THE GEODESIC FLOW AND GLOBAL GEOMETRY

by

Rafael Oswaldo Ruggiero

Dedicated to J. Palis, on his 60th. birthday

Abstract. — Let (M, g) be a compact Riemannian manifold without conjugate points. Suppose that the horospheres in $(\widetilde{M}, g)$ depend continuously on their normal directions. Then we show that geodesic rays diverge uniformly in the universal covering $(\widetilde{M}, g)$. We give some applications of this result to the study of the dynamics of the geodesic flow and the global geometry of manifolds without conjugate points.

Introduction

The problem of the divergence of geodesic rays in manifolds without conjugate points is one of the most natural, yet unsolved, questions of the theory. Recall that a C^∞ Riemannian, n-dimensional manifold (M, g) has no conjugate points if the exponential map is nonsingular at every point. The universal covering $\widetilde{M}$ of a manifold (M, g) is diffeomorphic to R^n, and the metric spheres in $(\widetilde{M}, g)$ — the universal covering endowed with the pullback of g — are diffeomorphic to the standard sphere in R^n. Given a point $p \in \widetilde{M}$, and two geodesics γ, β in $(\widetilde{M}, g)$ parametrized by arclength such that $p = \gamma(0) = \beta(0)$, we say that these geodesics diverge if $\lim_{t\to+\infty} d(\gamma(t), \beta(t)) = \infty$. Although two different geodesic rays starting from a point in $\widetilde{M}$ diverge in all well-known examples of manifolds without conjugate points (e.g., nonpositive curvature, no focal points, bounded asymptote), there is no general proof of this fact so far. The problem has been already considered by L. Green [**11**] in the late 50's, where Green deals with the divergence of radial Jacobi fields. Later, P. Eberlein [**6**] proves that radial Jacobi fields diverge along any geodesic in

2000 ***Mathematics Subject Classification.*** — 37D40, 53C23, 53C70.
Key words and phrases. — Conjugate points, horospheres, divergence of geodesic rays, local product structure, Gromov hyperbolic space, expansive flow.

Partially supported by CNPq, FAPERJ, and TWAS.

(M,g), but observes that the divergence might not be uniform, it could depend on the geodesic (in the same work [**6**], Eberlein points out a gap in Green's paper). The divergence of rays and Jacobi fields is related with many important geometric properties of manifolds without conjugate points, like the continuity of the horospherical foliations and Green bundles, and the existence of good compactifications of $\widetilde{M}$. This motivated somehow the introduction of some categories of manifolds without conjugate points in the literature (see for instance [**5**], [**6**], [**8**], for the so-called bounded asymptote condition, [**16**] for the Axiom of asymptoticity, [**14**] for the proof of the superlinear divergence of radial Jacobi fields in manifolds with bounded asymptote). The usual approach to the proofs of the continuity of horospheres, Green bundles, and divergence of rays, relies on strong assumptions on the asymptotic behaviour of geodesics and Jacobi fields (e.g., convexity in the case of nonpositive curvature; uniformly bounded asymptotic behaviour of Green Jacobi fields in the case of manifolds without focal points and manifolds with bounded asymptote). We shall present in this paper a more topological approach to the problem of the divergence of rays, based on simple variational properties of horospheres. Given $\theta = (p,v)$ in the unit tangent bundle $T_1\widetilde{M}$ of $\widetilde{M}$, we shall denote by $\gamma_\theta(t)$ the geodesic parametrized by arclength whose initial conditions are $\gamma_\theta(0) = p$, $\gamma'_\theta(0) = v$. We shall denote by $H_\theta(t)$ the horosphere of the geodesic γ_θ containing the point $\gamma_\theta(t)$. We say that the map $\theta \mapsto H_\theta(0)$ is continuous (in the compact open topology) if given a compact ball $B_r(q) \subset \widetilde{M}$ of radius r, and $\varepsilon > 0$, there exists $\delta = \delta(r,q,\varepsilon)$ such that if $\| \theta - \alpha \| \leqslant \delta$ then the Hausdorff distance d_H between the sets

$$d_H(H_\theta(0) \cap B_r(q), H_\alpha(0) \cap B_r(q)) \leqslant \varepsilon.$$

The introduction of this notion is motivated by the works of Pesin [**16**], Eschenburg [**8**], and Ballmann, Brin, and Burns [**1**]. Observe that, if M is compact, the number δ above does not depend on the point q, since every horosphere has an isometric image that meets a compact fundamental domain of $\widetilde{M}$ (horospheres are preserved by isometries of $(\widetilde{M},g)$). In all known examples of manifolds without conjugate points the map $\theta \mapsto H_\theta(0)$ is continuous. Moreover, the assumption of the continuity of horospheres does not carry (a priori) any restrictions on either the convexity of the metric or the behaviour of Jacobi fields. The main result of the paper is the following:

Theorem 1. — *Let (M,g) be a compact, C^∞ Riemannian manifold without conjugate points. Assume that the map $\theta \mapsto H_\theta(0)$ is continuous in $T_1\widetilde{M}$. Then, every two different geodesics $\gamma(t)$, $\beta(t)$ with $\gamma(0) = \beta(0)$ in $\widetilde{M}$ diverge.*

The proof of Theorem 1 is done in Sections 1 and 2, where we also study some general problems concerning asymptoticity properties of geodesics which were introduced by Croke and Schroeder in [**4**]. Namely, consider the relation $\boldsymbol{R}$ between geodesics in $\widetilde{M}$ defined by: $\gamma\ \boldsymbol{R}\ \beta$ if and only if γ is a Busemann asymptote of β. We show in

Section 1 that, under our continuity hypothesis, this relation is an equivalence relation. In the remaining sections we give some applications of Theorem 1. The results in Section 3 are inspired in the following classical result of Eberlein: Let (M, g) be a compact, C^∞ Riemannian manifold without conjugate points. Assume that the Green subbundles $E^s(\theta)$, $E^u(\theta)$ are linearly independent at every point $\theta \in T_1M$. Then the geodesic flow of (M, g) is Anosov. Recall that the geodesic flow $\phi_t : T_1M \to T_1M$ is defined by $\phi_t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$. We obtain in Section 3 a sort of topological version of Eberlein's result. Recall that $(\widetilde{M}, g)$ is a *Gromov hyperbolic space* if there exists $\delta > 0$ such that every geodesic triangle formed by the union of three geodesic segments $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_0]$ satisfies the following property: the distance from any $p \in [x_i, x_{i+1}]$ to $[x_{i+1}, x_{i+2}] \cup [x_{i+2}, x_i]$ is bounded above by δ (the indices are taken mod. 3). The main Theorem of Section 3 is the following.

Theorem 2. — *Let (M, g) be a compact Riemannian manifold without conjugate points. Suppose that the map $\theta \mapsto H_\theta(0)$ is continuous in the compact open topology in $\widetilde{M}$. Then, if $H_{(p,v)}(0) \cap H_{(p,-v)}(0) = \{p\}$ for every $(p, v) \in T_1\widetilde{M}$, the universal covering $(\widetilde{M}, g)$ is a Gromov hyperbolic space.*

Using some results in [**18**] we shall show that Theorem 2 is equivalent to the following result:

Theorem 3. — *Let (M, g) be a compact Riemannian manifold without conjugate points. Suppose that the canonical liftings in T_1M of the submanifolds $H_{(p,v)}(0)$, $H_{(p,-v)}(0)$ give rise to continuous foliations H^s, H^u having a local product structure. Then $(\widetilde{M}, g)$ is a Gromov hyperbolic space.*

For the definition of the canonical liftings of the horospheres we refer to Section 3. A pair of ϕ_t-invariant foliations F_1, F_2 in T_1M has a *local product structure* if there exists an atlas $\{\Phi_i : U_i \subset T_1M \to R^{2n-1}\}$ of T_1M such that

(1) Every Φ_i is continuous.

(2) Each local chart is of the form $\Phi_i = (x^i, y^i, t)$, $t \in (-\varepsilon, \varepsilon)$, where the level sets $x^i = constant$, $y^i = constant$ are connected components of the foliations F_1, F_2 respectively.

In virtue of Theorems 2 and 3, we can say that the topological transversality (meaning local product structure) of the horospherical foliations in T_1M implies that $\widetilde{M}$ is a Gromov hyperbolic space. Notice that Theorem 1 is true for manifolds of nonpositive curvature, because the hypotheses in the Theorem imply that there are no flat planes in $\widetilde{M}$ ([**5**]). It also holds for manifolds without focal points, but if we allow focal points many key facts of the theory (convexity, bounded asymptotic behaviour of Jacobi fields and geodesics, etc.) might not hold.

In Section 4 we get some results concerning the boundary of a Gromov hyperbolic group that covers a compact manifold without conjugate points. Suppose that the map $\theta \mapsto H_\theta(0)$ is continuous. Then we show that, if the fundamental group of M is Gromov hyperbolic, its ideal boundary is a sphere. This fact is well known for compact manifolds of nonpositive curvature whose fundamental group is Gromov hyperbolic. However, if we drop the assumption on the curvature it is not clear whether the ideal boundary of the fundamental group is a sphere.

Finally, in Section 5, we apply the results of Sections 1, 2 to manifolds satisfying the so-called Axiom of Asymptoticity, introduced by Pesin [**16**]. This notion is perhaps the first one in the literature of the research about continuity of horospheres which does not consider any assumptions on the C^2 features of the metric (convexity, Jacobi fields).

1. Horospheres and Busemann flows in $\widetilde{M}$

Throughout the paper, (M, g) will be a C^∞, compact Riemannian manifold without conjugate points. All the geodesics will be parametrized by arc length. We shall often call by $[p, q]$ the geodesic segment joining two points in $\widetilde{M}$. A very special property of manifolds with no conjugate points is the existence of the so-called *Busemann functions*: given $\theta = (p, v) \in T_1\widetilde{M}$ the *Busemann function* $b^\theta : \widetilde{M} \to R$ associated to θ is defined by

$$b^\theta(x) = \lim_{t\to+\infty}(d(x, \gamma_\theta(t)) - t)$$

The level sets of b^θ are the *horospheres* $H_\theta(t)$ where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$. Notice that $\gamma_\theta(t)$ intersects each level set of b^θ perpendicularly at only one point in $H_\theta(t)$, and that $b^\theta(H_\theta(t)) = -t$ for every $t \in R$. Next, we list some basic properties of horospheres and Busemann functions that will be needed in the forthcoming sections (see [**16**], [**4**] for instance, for details).

Lemma 1.1

(1) *b^θ is a C^1 function for every θ.*

(2) *The gradient ∇b^θ has norm equal to one at every point.*

(3) *Every horosphere is a C^{1+K}, embedded submanifold of dimension $n-1$ (C^{1+K} means K-Lipschitz normal vector field), where K is a constant depending on curvature bounds.*

(4) *The orbits of the integral flow of $-\nabla b^\theta$, $\psi_t^\theta : \widetilde{M} \to \widetilde{M}$, are geodesics which are everywhere perpendicular to the horospheres H_θ. In particular, the geodesic γ_θ is an orbit of this flow and we have that*

$$\psi_t^\theta(H_\theta(s)) = H_\theta(s + t)$$

for every $t, s \in R$.

A geodesic β is *asymptotic* to a geodesic γ in $\widetilde{M}$ if there exists a constant $C > 0$ such that $d(\beta(t), \gamma(t)) \leqslant C$ for every $t \geqslant 0$. We shall denote by *Busemann asymptotes* of γ_θ