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# COMPLEX SCHOTTKY GROUPS

*by*

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***Abstract.*** — In this work we study a certain type of discrete groups acting on higher dimensional complex projective spaces. These generalize the classical Schottky groups acting on the Riemann sphere. We study the limit sets of these actions, which turn out to be solenoids. We also look at the compact complex manifols obtained as quotient of the region of discontinuity, divided by the action. We determine their topology and the dimension of the space of their infinitesimal deformations. We show that every such deformation arises from a deformation of the embedding of the group in question into the group of automorphisms of the corresponding complex projective space, which is a reminiscent of the classical Teichmüller theory.

## Introduction

The theory of Kleinian groups introduced by Poincaré [**Po**] in the 1880's played a major role in many parts of mathematics throughout the 20th century, as for example in Riemann surfaces and Teichmüller theory, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry, 3-manifolds theory, etc. These groups are, by definition, discrete groups of holomorphic automorphisms of the complex projective line $P^1_{\mathbb{C}}$, whose limit set is not the whole $P^1_{\mathbb{C}}$. Equivalently, these can be regarded as groups of isometries of the hyperbolic 3-space, or as groups of conformal automorphisms of the sphere $S^2$. Much of the theory of Kleinian groups has been generalised to conformal Kleinian groups in higher dimensions (also called

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*Möbius* or *hyperbolic* Kleinian groups), *i.e.*, to discrete groups of conformal automorphisms of the sphere $S^n$ whose limit set is not the whole sphere (see, for instance, [**Ku1, Ku2, Ma1, Su1, Su2, Su3, Su4**]).

Many interesting results about the dynamics of rational maps on $P^1_{\mathbb{C}}$ in the last decades have been motivated by the dynamics of Kleinian groups, and there is an interesting "dictionary" between these two theories (see, for instance, [**Su1, Su2, Su3, Su4, Mc1, Mc2**]). The theory of rational maps has also been generalised to automorphisms of $P^2_{\mathbb{C}}$, and recently many results are being obtained about the dynamics of automorphisms and rational endomorphisms of $P^n_{\mathbb{C}}$ in general. This led us to define in [**SV**] the concept of a *higher dimensional complex Kleinian groups*. By this we meant (infinite) discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, the group of *holomorphic* automorphisms of $P^n_{\mathbb{C}}$, $n > 1$, acting with a non-empty region of discontinuity.

One of the most interesting families of (conformal) Kleinian groups is provided by the Schottky groups, and the aim of this article is to study the analogous construction for groups acting by holomorphic transformations on complex projective spaces. We call these *Complex Schottky Groups*.

We consider an arbitrary configuration $\{(L_1, M_1), \dots, (L_r, M_r)\}$ of pairs of projective $n$-spaces in $P^{2n+1}_{\mathbb{C}}$, which are all of them pairwise disjoint. Given arbitrary neighbourhoods $U_1, \dots, U_r$ of the $L_i$'s, pairwise disjoint, we show that there exists, for each $i = 1, \dots, r$, projective transformations $T_i$ of $P^{2n+1}_{\mathbb{C}}$, which interchange the interior with the exterior of a compact tubular neighbourhood $N_i$ of $L_i$ contained in $U_i$, leaving invariant the boundary $E_i = \partial(N_i)$. The $E_i$'s are *mirrors*, they play the same role in $P^{2n+1}_{\mathbb{C}}$ as circles play in $S^2$ to define the classical Schottky groups. Each mirror $E_i$ is a $(2n+1)$-sphere bundle over $P^n_{\mathbb{C}}$. The group of automorphisms of $P^{2n+1}_{\mathbb{C}}$ generated by the $T_i$'s is a complex Kleinian group $\Gamma$. The region of discontinuity $\Omega(\Gamma)$ is a fibre bundle over $P^n_{\mathbb{C}}$ with fibre $S^{2n+2}$ minus a Cantor set $\mathcal{C}$. The limit set $\Lambda$ is the complement of $\Omega(\Gamma)$ in $P^{2n+1}_{\mathbb{C}}$; it is the set of accumulation points of the $\Gamma$-orbit of the $L_i' s$, and it is a product $\mathcal{C} \times P^n_{\mathbb{C}}$. The action of $\Gamma$ on this set of projective lines is minimal in the sense that the $\Gamma$-orbit of every point $x_o$ in $P^{2n+1}_{\mathbb{C}}$ accumulates to (at least a point in) each one of the projective lines in $\Lambda$. This set is transversally *projectively self-similar*, i.e., $\Lambda$ corresponds to a Cantor set in the Grassmannian $G_{2n+1,n}$, which is dynamically-defined. Hence $\Lambda$ is a *solenoid* (or *lamination*) by projective spaces, which is transversally Cantor and projectively self-similar. Each of these groups $\Gamma$ contains a subgroup $\check{\Gamma}$ of index two, which is a free group of rank $r-1$ and acts *freely* on $\Omega(\Gamma)$. The quotient $\Omega(\Gamma)/\check{\Gamma}$ is a compact complex manifold, which is a fibre bundle over $P^n_{\mathbb{C}}$ with fibre the connected sum of $(r-1)$ copies of $S^{2n+1} \times S^1$. As mentioned above, these manifolds have a canonical projective structure [**Gu**], *i.e.*, they have an atlas $\{(\mathcal{U}_i, \phi_i)\}$ whose changes of coordinates are restrictions of complex projective transformations. However, these manifolds are never Kähler, due to cohomological reasons. When $n = 1$, the manifolds that we obtain are *Pretzel twistor spaces* in the sense of [**Pe**]; and if the configuration $\{(L_1, M_1), \ldots, (L_r, M_r)\}$ consists of twistor lines of the fibration $p\colon P^3_{\mathbb{C}} \to S^4$, then $\Gamma$ and $\check{\Gamma}$ descend to conformal Schottky groups on $S^4$. In this case $\Omega(\Gamma)/\check{\Gamma}$ is the twistor space of the conformally flat manifold $S^4/p(\check{\Gamma})$, which is a Schottky manifold [**Ku2**]; $\Omega(\Gamma)/\check{\Gamma}$ is a *flat twistor space* [**Si**]. We also generalise our construction of Schottky groups to $P^{\infty}_{\mathbb{C}}$, the projectivization of a separable complex infinite dimensional Hilbert space.

We then compare the deformations of our Schottky groups with the deformations of the complex manifolds that one gets as quotients of the action of the group on its region of discontinuity. For this we estimate an upper bound for the Hausdorff dimension of the limit set of the complex Schottky groups. We use this to show that, with the appropriate conditions for the Schottky group $\check{\Gamma}$, the Kuranishi space $\mathfrak{K}$ of versal deformations of the complex manifold $M_{\check{\Gamma}} = \Omega(\check{\Gamma})/\check{\Gamma}$, is smooth near the reference point determined by $M_{\check{\Gamma}}$. Furthermore, we estimate the dimension of $\mathfrak{K}$ and we prove that every infinitesimal deformation of $M_{\check{\Gamma}}$ actually corresponds to an infinitesimal deformation of the group $\check{\Gamma}$ in the projective group $\mathrm{PSL}(2n+2, \mathbb{C})$, in analogy with the classical Teichmüller and moduli theory for Riemann surfaces.

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## 1. Complex Schottky groups

We recall that (in the classical case) the Schottky groups are obtained by considering pairwise disjoint $(n-1)$-spheres $\mathbb{S}_1, \ldots, \mathbb{S}_r$ in $S^n$, see [**Ma2**]. Each sphere $\mathbb{S}_i$ plays the role of a mirror: it divides $S^n$ in two diffeomorphic components, and one has an involution $T_i$ of $S^n$ interchanging these components, the inversion on $\mathbb{S}_i$. The Schottky group is defined to be the group of conformal transformations generated by these involutions. We are going to make a similar construction on $P^{2n+1}_{\mathbb{C}}$, $n > 0$. (For $n = 0$, if we take $P^0_{\mathbb{C}}$ to be a point, this construction gives the classical Schottky groups on $P^1_{\mathbb{C}}$.)

Consider the subspaces of $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ defined by $\widehat{L}_0 := \{(a, 0) \in \mathbb{C}^{2n+2}\}$ and $\widehat{M}_0 := \{(0, b) \in \mathbb{C}^{2n+2}\}$. Let $\widehat{S}$ be the involution of $\mathbb{C}^{2n+2}$ defined by $\widehat{S}(a, b) = (b, a)$. This interchanges $\widehat{L}_0$ and $\widehat{M}_0$.

***1.1. Lemma***. — *Let* $\Phi\colon \mathbb{C}^{2n+2} \to \mathbb{R}$ *be given by* $\Phi(a, b) = |a|^2 - |b|^2$. *Then:*

i) $\widehat{E}_{\widehat{S}} := \Phi^{-1}(0)$ *is a real algebraic hypersurface in* $\mathbb{C}^{2n+2}$ *with an isolated singularity at the origin* 0. *It is embedded in* $\mathbb{C}^{2n+2}$ *as a (real) cone over* $S^{2n+1} \times S^{2n+1}$, *with vertex at* $0 \in \mathbb{C}^{2n+2}$.

ii) $\widehat{E}_{\widehat{S}}$ *is invariant under multiplication by* $\lambda \in \mathbb{C}$, *so it is in fact a complex cone.* $\widehat{E}_{\widehat{S}}$ *separates* $\mathbb{C}^{2n+2} - \{(0,0)\}$ *in two diffeomorphic connected components* $U$ *and* $V$, *which contain respectively* $\widehat{L_0} - \{(0,0)\}$ *and* $\widehat{M_0} - \{(0,0)\}$. *These two components are interchanged by the involution* $\widehat{S}$, *for which* $\widehat{E}_{\widehat{S}}$ *is an invariant set.*

iii) *Every linear subspace* $\widehat{K}$ *of* $\mathbb{C}^{2n+2}$ *of dimension* $n+2$ *containing* $\widehat{L_0}$ *meets transversally* $\widehat{E}_{\widehat{S}}$ *and* $\widehat{M_0}$. *Therefore a tubular neighbourhood* $V$ *of* $\widehat{M_0} - \{(0,0)\}$ *in* $P_{\mathbb{C}}^{2n+1}$ *is obtained, whose normal disc fibres are of the form* $\widehat{K} \cap V$, *with* $\widehat{K}$ *as above.*

*Proof.* — The first statement is clear because $\Phi$ is a quadratic form with $0 \in \mathbb{C}^{2n+2}$ as unique critical point. Clearly $\widehat{E}_{\widehat{S}}$ is invariant under multiplication by complex numbers, so it is a complex cone. That $\widehat{E}_{\widehat{S}} \cap S^{4n+3} = S^{2n+1} \times S^{2n+1} \subset \mathbb{C}^{2n+2}$, is because this intersection consists of all pairs $(x,y)$ so that $|x| = |y| = 1/\sqrt{2}$. That $\widehat{S}$ leaves $\widehat{E}_{\widehat{S}}$ invariant is obvious, and so is that $\widehat{S}$ interchanges the two components of $\mathbb{C}^{2n+2} - \{(0,0)\}$ determined by $\widehat{E}_{\widehat{S}}$, which must be diffeomorphic because $\widehat{S}$ is an automorphism. Finally, if $\widehat{K}$ is a subspace as in the statement (iii), then $\widehat{K}$ meets transversally $\widehat{E}_{\widehat{S}}$, because through every point in $\widehat{E}_{\widehat{S}}$ there exists an affine line in $\widehat{K}$ which is transverse to $\widehat{E}_{\widehat{S}}$. □

Let $S$ be the linear projective involution of $P_{\mathbb{C}}^{2n+1}$ defined by $\widehat{S}$. Since $\widehat{E}_{\widehat{S}}$ is a complex cone, it projects to a codimension 1 real submanifold of $P_{\mathbb{C}}^{2n+1}$, that we denote by $E_S$.

***1.2. Corollary***

i) $E_S$ *is an invariant set of* $S$.

ii) $E_S$ *is a* $S^{2n+1}$*-bundle over* $P_{\mathbb{C}}^n$, *in fact* $E_S$ *is the sphere bundle associated to the holomorphic bundle* $(n+1)\mathcal{O}_{P_{\mathbb{C}}^n}$, *which is the normal bundle of* $P_{\mathbb{C}}^n$ *in* $P_{\mathbb{C}}^{2n+1}$.

iii) $E_S$ *separates* $P_{\mathbb{C}}^{2n+1}$ *in two connected components which are interchanged by* $S$ *and each one is diffeomorphic to a tubular neighbourhood of the canonical* $P_{\mathbb{C}}^n$ *in* $P_{\mathbb{C}}^{2n+1}$.

***Definition***. — We call $E_S$ *the canonical mirror* and $S$ *the canonical involution.*

It is an exercise to show that (1.1) holds in the following more generally setting. Of course one has the equivalent of (1.2) too.

***1.3. Lemma***. — *Let* $\lambda$ *be a positive real number and consider the involution*

$$\widehat{S}_\lambda : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{n+1},$$

*given by* $\widehat{S}_\lambda(a,b) = (\lambda b, \lambda^{-1} a)$. *Then* $\widehat{S}_\lambda$ *also interchanges* $\widehat{L}_0$ *and* $\widehat{M}_0$, *and the set*

$$\widehat{E}_\lambda = \{(a,b) : |a|^2 = \lambda^2 |b|^2\}$$

*satisfies, with respect to* $\widehat{S}_\lambda$, *the analogous properties* (i)-(iii) *of (1.1) above.*