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STABLE ACCESSIBILITY IS C^1 DENSE

by

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Abstract. — We prove that in the space of all C^r ($r \geqslant 1$) partially hyperbolic diffeomorphisms, there is a C^1 open and dense set of accessible diffeomorphisms. This settles the C^1 case of a conjecture of Pugh and Shub. The same result holds in the space of volume preserving or symplectic partially hyperbolic diffeomorphisms. Combining this theorem with results in [**Br**], [**Ar**] and [**PugSh3**], we obtain several corollaries. The first states that in the space of volume preserving or symplectic partially hyperbolic diffeomorphisms, topological transitivity holds on an open and dense set. Further, on a symplectic n-manifold ($n \leqslant 4$) the C^1-closure of the stably transitive symplectomorphisms is precisely the closure of the partially hyperbolic symplectomorphisms. Finally, stable ergodicity is C^1 open and dense among the volume preserving, partially hyperbolic diffeomorphisms satisfying the additional technical hypotheses of [**PugSh3**].

Introduction

This paper is about the accessibility property of partially hyperbolic diffeomorphisms. We show that accessibility holds for a C^1 open and dense set in the space of all partially hyperbolic diffeomorphisms, thus settling the C^1 version of a conjecture of Pugh and Shub [**PugSh1**]. Partially hyperbolic diffeomorphisms are similar to Anosov diffeomorphisms, in that they possess invariant hyperbolic directions, but unlike Anosov diffeomorphisms, they can also possess invariant directions of non-hyperbolic behavior. Accessibility means that the hyperbolic directions fill up the manifold on a macroscopic scale. Accessibility often provides enough hyperbolicity for a variety of chaotic properties, such as topological transitivity [**Br**] and ergodicity

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[**PugSh3**], to hold. As a consequence, we derive several density results about stable ergodicity and stable transitivity among partially hyperbolic diffeomorphisms.

Let M be a smooth compact, connected and boundaryless Riemannian manifold. A diffeomorphism $f : M \to M$ is *partially hyperbolic* if the tangent bundle to M splits as a Tf-invariant sum

$$TM = E^u \oplus E^c \oplus E^s,$$

such that Tf uniformly expands all vectors in E^u and uniformly contracts all vectors in E^s, while vectors in E^c are neither contracted as strongly as any vector in E^s nor expanded as strongly as any vector in E^u. More precisely, for each $p \in M$, there exist $0 < a_p < b_p < 1 < B_p < A_p$ such that:

$$\|T_p f|_{E^s}\| \leqslant a_p < b_p \leqslant m(T_p f|_{E^c}) \leqslant \|T_p f|_{E^c}\| \leqslant B_p < A_p \leqslant m(T_p f|_{E^u}),$$

where $m(T) = \|T^{-1}\|^{-1}$. Throughout this paper we assume that both subbundles E^u and E^s are nontrivial.

A more stringent condition, often called partial hyperbolicity in the literature (cf. [**BrPe**], [**BuPuShWi**]) requires that the constants a_p, b_p, A_p and B_p be chosen independent of p. Since the results in this paper apply to diffeomorphisms satisfying the weaker condition, to avoid excessive terminology, we will use the term partial hyperbolicity in the broader sense.

A partially hyperbolic diffeomorphism f is *accessible* if, for every pair of points $p, q \in M$, there is a C^1 path from p to q whose tangent vector always lies in $E^u \cup E^s$ and vanishes at most finitely many times. We say f is *stably accessible* if every g sufficiently C^1-close to f is accessible. We prove here the following theorem.

Main Theorem. — *For any $r \geqslant 1$, stable accessibility is C^1 dense among the C^r, partially hyperbolic diffeomorphisms of M, volume preserving or not. If M is a symplectic manifold, then stable accessibility is C^1 dense among C^r, symplectic partially hyperbolic diffeomorphisms of M.*

Related to the Main Theorem is the result of Niţică and Török [**NiTö**] that stable accessibility is C^r-dense among partially hyperbolic diffeomorphisms with 1-dimensional, integrable center bundle E^c. Other results about stable accessibility treat more special classes of diffeomorphisms, such as time-one maps of Anosov flows [**BuPuWi**], skew products [**BuWi1**], certain systems where $E^u \oplus E^s$ is integrable [**ShWi**], and systems whose partially hyperbolic splitting is C^1 [**PugSh2**].

The Main Theorem has several corollaries. The first corollary concerns the topological transitivity of partially hyperbolic diffeomorphisms and follows immediately from a theorem of Brin [**Br**]. Denote by $\mathcal{PH}^r(M)$ the set of C^r partially hyperbolic diffeomorphisms of M. If μ and ω are, respectively, Riemannian volume and a symplectic form on M, then set

$$\mathcal{P}H^r_\mu(M) = \{f \in \mathcal{P}H^r(M) \mid f_*(\mu) = \mu\}, \text{ and}$$
$$\mathcal{P}H^r_\omega(M) = \{f \in \mathcal{P}H^r(M) \mid f^*(\omega) = \omega\}.$$

Corollary 0.1. — *For $r \geqslant 1$, there is a C^1-open and dense set of topologically transitive diffeomorphisms in $\mathcal{P}H^r_\mu(M)$. If M has a symplectic form ω, then there is a C^1-open and dense set of transitive diffeomorphisms in $\mathcal{P}H^r_\omega(M)$.*

This corollary is false without the volume preservation assumption. Niţică and Török have shown in [**NiTö**] that there is an open set of accessible non-transitive diffeomorphisms. While it is plausible that for a C^1 open and dense set of diffeomorphisms in the space $\mathcal{P}H^r(M)$, there are only finitely many transitivity components, it is not a direct corollary of the Main Theorem.

M.-C. Arnaud has shown in [**Ar**] that if M is a symplectic 4-manifold, then the stably transitive diffeomorphisms in $\mathrm{Diff}^r_\omega(M)$ are partially hyperbolic. (The same result has been announced by J. Xia in arbitrary dimension). Hence there is a complete picture in dimension 4 of the stably transitive diffeomorphisms, which we summarize in the next corollary.

Corollary 0.2. — *Let M be a symplectic manifold with $\dim(M) \leqslant 4$. The C^1-closure of the stably transitive diffeomorphisms in $\mathrm{Diff}^r_\omega(M)$ coincides with the C^1 closure of the partially hyperbolic ones.*

In other words, invariant tori are essentially the only obstacle for topological transitivity in the symplectic category, at least if $\dim(M) \leqslant 4$. We conjecture that the same is true in the volume preserving case.

Conjecture 0.3. — *In the space of volume preserving diffeomorphisms, the C^1-closure of the stably transitive diffeomorphims coincides with the closure of the diffeomorphisms admitting a dominated splitting.*

For a discussion of the dominated splitting condition and some results related to Conjecture 0.3 see [**Vi**]. Even though the results of this paper could be useful in attacking this conjecture some other ideas (possibly ones from the paper [**BonDi**]) are necessary to solve this problem. Here we note only that in [**BV**] a volume preserving example is presented which is stably transitive yet not partially hyperbolic. A. Tahzhibi has announced a proof that these example are in fact stably ergodic.

Another corollary of the Main Theorem concerns ergodicity of $f \in \mathcal{P}H^r_\mu(M)$. Pugh and Shub proved the following theorem:

Theorem 0.4 ([**PugSh3**, Theorem A]). — *Let $f \in \mathcal{P}H^2_\mu(M)$. If f is center bunched, dynamically coherent, and essentially accessible, then f is ergodic.*

Thus we also have the corollary:

Corollary 0.5. — *Among the center bunched, stably dynamically coherent diffeomorphisms in $\mathcal{PH}^2_\mu(M)$, stable ergodicity is C^1 open and dense.*

Theorem 0.4 refers to partially hyperbolic diffeomorphisms in the stronger sense described earlier, but recently Burns and Wilkinson [**BuWi2**] have shown that these results extend to the larger class of partially hyperbolic diffeomorphisms described in this paper (satisfying additional center bunching conditions). For a description of examples of diffeomorphisms satisfying the conditions "center bunched" and "stably dynamically coherent" see the survey paper [**BuPuShWi**]. In particular, the corollary implies that there is a C^1-open neighborhood $\mathcal{U} \subset \mathcal{PH}^2_\mu(M)$ of f in which stable ergodicity is C^1-open and dense, where f is the time-t map of an Anosov flow, a compact group extension of an Anosov diffeomorphism, an ergodic automorphism of a torus or nilmanifold, or a partially hyperbolic translation on a compact homogeneous space.

This paper arose out of an attempt to prove the following conjecture of Pugh and Shub.

Conjecture 0.6 **([PugSh2**, Conjecture 4] **and [PugSh3**, Conjecture 2]**)**
Stable accessibility is C^r - dense in both $\mathcal{PH}^r(M)$ and $\mathcal{PH}^r_\mu(M)$.

In the spirit of Theorem 0.4, Pugh and Shub also conjectured:

Conjecture 0.7 **([PugSh3**, Conjecture 3]**)**. — *A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.*

Finally, combining Conjectures 0.6 and 0.7, they conjectured:

Conjecture 0.8 **([PugSh3**, Conjecture 4]**)**. — *Stable ergodicity is C^r-dense in $\mathcal{PH}^r_\mu(M)$.*

As with Theorem 0.4, these conjectures refer to the narrower class of partially hyperbolic diffeomorphisms described above, but in light of the results in [**BuWi2**] and this paper, it seems reasonable to extend them to the class under consideration here.

The question of accessibility is closely related to problems in control theory (see, e.g. [**Lo**]). In fact, analogous density theorems in control theory initially suggested the Conjectures 0.6 and 0.7. The sole reason that the results in control theory cannot be directly transported to this setting is that we do not perturb the bundles E^u and E^s directly, but rather the diffeomorphism f. We'd like to be able to say that a specific perturbation of f has a specific effect on E^u and E^s. What makes this difficult is that $E^s(p)$ and $E^u(p)$ are determined by the entire forward and backward orbit of p, respectively; a perturbation will have various effects along the length of this orbit, some desirable and others not.

The key observation that permits a measure of control is that the effects of the perturbation are greatest along the first few iterates of p. To maximize our control