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ANOSOV GEODESIC FLOWS FOR EMBEDDED SURFACES

by

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Abstract. — In this paper we embed a high genus surface in $\mathbb{R}^3$ so that its geodesic flow has no conjugate points and is Anosov, despite the fact that its curvature cannot be everywhere negative.

1. Introduction

At the International Conference on Dynamical Systems held in Rio de Janeiro in July, 2000, Michael Herman asked whether the geodesic flow for an embedded surface in $\mathbb{R}^3$ can be uniformly hyperbolic, i.e., Anosov. Using techniques from our paper [**5**] and a suggestion of John Franks and Clark Robinson, we answer Herman's question affirmatively. The embedded surface looks like a spherical shell with many holes drilled through it. See Figures 1 and 2.

The Lobachevsky-Hadamard Theorem states that if a Riemann manifold has negative sectional curvature then its geodesic flow is Anosov. The celebrated thesis of Anosov [**1**] shows that this implies ergodicity, in fact the Bernoulli property, a stronger form of ergodicity.

In [**2**], Burns and Donnay showed that every surface M embeds in $\mathbb{R}^3$ so that its geodesic flow is Bernoulli; however, this cannot be a consequence of M having negative curvature. For a compact surface $M \subset \mathbb{R}^3$ necessarily has regions of positive curvature, the standard explanation being that there is a smallest sphere S which contains M, and there are points at which S is tangent to M. At these points the

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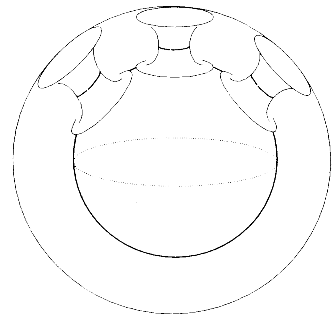

FIGURE 1. An embedded surface formed by connecting two concentric spheres with many tubes.

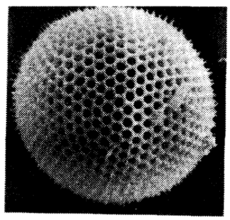

FIGURE 2. The radiolarian *Aulonia hexagona*, a marine micro-organism, as it appears through an electron microscope, by S.A. Kling.

curvature of M is positive. By continuity, the curvature of M is positive at nearby points too. The Bernoulli geodesic flows constructed by Burns and Donnay employ "focusing caps" to control the positive curvature. However, the caps are bounded by closed geodesics on which the curvature is zero, preventing uniform hyperbolicity. If the caps are perturbed to destroy these parabolic orbits the system can become non-ergodic [**3, 4**].

Instead of using caps, we use tubes of negative curvature together with the notion of a finite horizon geometry, which we introduced in [**5**], and are thereby able to show

Theorem A. — *There exist embedded surfaces in $\mathbb{R}^3$ for which the geodesic flows are Anosov.*

As an extension of Theorem A we discuss the immersed case, which has interest when the surface is not orientable.

Theorem B. — *There exist immersed non-orientable surfaces in $\mathbb{R}^3$ for which the geodesic flows are Anosov.*

The basic ingredient in our construction is illustrated in Figure 3; connect two flat tori (they are not embedded in $\mathbb{R}^3$) via a tube of negative curvature. The geodesic flow for this genus two surface is Bernoulli but not uniformly hyperbolic - since there

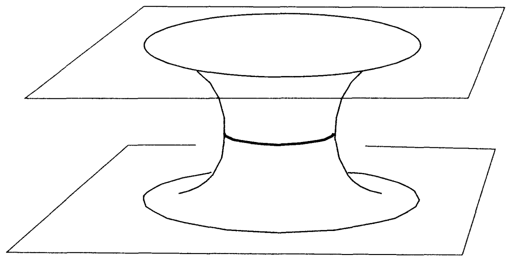

FIGURE 3. Two flat tori joined by a negatively curved tube.

are periodic geodesics lying completely in a flat region. If we now connect the two tori by enough tubes to produce a finite horizon pattern (see Section 2), i.e. every geodesic enters a tube in a bounded time, then the geodesic flow for this high genus surface is Anosov. To make an embedded Anosov example, we follow the suggestion of Franks and Robinson: reproduce the construction using very large and nearly flat concentric spheres instead of tori, again in a finite horizon pattern of tubes.

Remark. — Theorems A and B give the existence of high genus surfaces in $\mathbb{R}^3$ with Anosov geodesic flows, but we do not know a good lower bound on the genus. In [**6**], Wilhelm Klingenberg shows that no surface whose Riemann structure has conjugate points, which are produced by a surfeit of positive curvature, can have an Anosov geodesic flow. Hence our construction also provides examples of embedded surfaces without conjugate points. By Klingenberg's result, the sphere and torus never have Riemann structures whose geodesic flows are Anosov. So in particular, these surfaces cannot embed in $\mathbb{R}^3$ in such a way that their geodesic flows are Anosov. But what about the bitorus? Can it embed in $\mathbb{R}^3$ so that its geodesic flow is Anosov? Is it at least possible to embed the bitorus so that its metric has no conjugate points?

2. Finite Horizon

Let M be a surface equipped with a Riemann structure. A family $\mathcal{C}$ of curves $C_1, \ldots, C_k$ in M gives M *ϕ-finite horizon* if every unit length geodesic crosses at least one curve in $\mathcal{C}$ at an angle $\geqslant \phi$. In [**5**] we show in detail how to choose $\mathcal{C}$ that gives M finite horizon, when M is a surface embedded in $\mathbb{R}^3$ and its Riemann structure is the one it inherits from the embedding. Here is an outline of the construction.

We first construct a fine, smooth triangulation of M whose triangles have uniformly bounded eccentricity and nearly geodesic edges. (The eccentricity of a triangle is the reciprocal of its smallest vertex angle.) We then draw small geodesic discs at the vertices of the triangulation, and a string of N "pearl discs" along each edge of the triangulation outside the vertex discs. Finally, we draw $2N+2$ "wing discs" parallel to the string of pearl discs. Altogether this gives $9(N+1)$ discs per triangle. The pearl and wing discs have radius r, which is much less than the radius R of the vertex discs, and this makes the pearl and wing discs along one edge of a triangle disjoint from those along a different edge.

Technically, once we have a bound on the eccentricity of the triangles that appear in our triangulations, we choose R and N. We then keep R and N fixed, while we dilate the surface by a factor of 2^n, $n \to \infty$, making ever finer triangulations of the dilated surface that have nearly linear triangles of roughly unit size. The radii r of the pearl and wing disks vary depending on the length of the edge of the triangle but lie in a compact interval.

With respect to the flat Riemann structure, the disc pattern for a triangle is shown in Figure 4. Every unit segment starting inside the flat triangle must cross the bound-

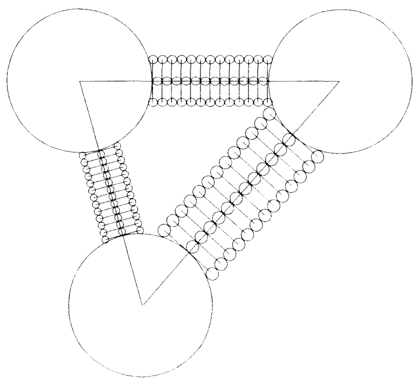

FIGURE 4. The pattern of discs for a linear triangle that gives the finite horizon property.

ary circles of these discs at some positive angle. By compactness, they cross at some uniformly positive angle ϕ, a fact that remains true under small perturbations. For example, if we shrink all the discs by a factor $\mu < 1$, where $1-\mu$ is small, they still give the finite horizon property for unit segments. Similarly, the finite horizon property still holds if the flat metric is replaced by a nearly flat metric.

Denote by $2^n M$ the surface gotten by dilating M by a factor 2^n. The Riemann structure of $2^n M$ restricted to a nearly linear triangle T of roughly unit size is nearly flat. Thus, the geodesic discs of radius μr and μR laid down in the pattern of Figure 4 are disjoint and give the finite horizon property for unit geodesics on $2^n M$ when n is large.

We then flatten these disjoint geodesic discs by pushing each into the tangent plane at its center. Slightly smaller round discs lie in the flattened geodesic discs, and they still give the finite horizon property. The net effect is that the given surface M is replaced by a new one, $2^n M$, with diameter roughly 2^n, and having a great number