ROBERTO FERNÁNDEZ
ANDRÉ TOOM

Non-Gibbsianness of the invariant measures of non-reversible cellular automata with totally asymmetric noise

Astérisque, tome 287 (2003), p. 71-87
<http://www.numdam.org/item?id=AST_2003__287__71_0>

Astérisque
287, 2003, p. 71–87

NON-GIBBSIANNESS OF THE INVARIANT MEASURES OF NON-REVERSIBLE CELLULAR AUTOMATA WITH TOTALLY ASYMMETRIC NOISE

by

Roberto Fernández & André Toom

Abstract. — We present a class of random cellular automata with multiple invariant measures which are all non-Gibbsian. The automata have configuration space $\{0,1\}^{\mathbb{Z}^d}$, with $d > 1$, and they are noisy versions of automata with the "eroder property". The noise is totally asymmetric in the sense that it allows random flippings of "0" into "1" but not the converse. We prove that all invariant measures assign to the event "a sphere with a large radius L is filled with ones" a probability μ_L that is too large for the measure to be Gibbsian. For example, for the NEC automaton $(-\ln \mu_L) \asymp L$ while for any Gibbs measure the corresponding value is $\asymp L^2$.

1. Introduction

Studies of cellular automata and of their continuous-time counterpart, the spin-flip dynamics, have been successful in determining how many invariant measures the automaton or dynamics have. Much less is known about properties of these measures. A natural question is whether they are Gibbsian, that is whether they could correspond to measures describing the equilibrium state of some statistical mechanical system. There are two categories of evolutions —both with local and strictly positive updating rates— for which the answer is known to be positive: (1) If the updating prescription has a high level of stochasticity —*high noise regime*—, in which case Gibbsianness comes together with uniqueness of the invariant measure [**15, 19, 18**]; and (2) if the updating satisfies a detailed balance condition for some Boltzmann-Gibbs weights [**20**]. Known cases of non-Gibbsianness, on the other hand, refer to automata where the updating rates are either non-strictly positive [**16**], [**30**, Chapter 7] or non-local [**23**].

2000 Mathematics Subject Classification. — 37B15, 37C40, 60A10.
Key words and phrases. — Gibbs vs. non-Gibbs measures, cellular automata, invariant measures, non-ergodicity, eroders, convex sets.

R.F.: Travel support by an agreement USP-Cofecub (projet UC/68/99, *Comportement à temps intermédiaire des systèmes aléatoires*).
A.T.: Supported by FAPESP, grant # 98/15994-0 and CNPq, grant # 300991/1998-3.

In this paper we present some examples of stochastic *non-reversible automata* —that is, automata not satisfying any form of detailed balance—, with multiple invariant measures, all of them non-Gibbsian. Our class of automata can be seen as a generalization of the North-East-Center (NEC) majority model introduced in [**24**] and discussed in many papers. Its non-ergodicity was first proved in [**28**] (see also the discussion in [**15**]) and later by another method in [**2**]. Also it was simulated more than once [**1, 21, 22**]. Models of this sort are obtained by superimposing stochastic errors (noise) to deterministic automata having the so-called *eroder property*: finite islands of aligned spins, within a sea of spins aligned in the opposite direction, disappear in a finite time.

We allow only *one-sided* noise or stochastic error —a "0" can stochastically be turned into a "1", but not the reverse. Thus some of our transition rates are zeros and therefore the "dichotomy" result of [**20**, Corollary 1] is not applicable. Our work does not settle the long-standing issue of the Gibbsianness of the invariant measures of NEC models with non totally asymmetric noise. There are conflicting arguments and evidences for the model with symmetric noise: An interesting heuristic argument has been put forward [**30**, Chapter 5] pointing in the direction of Gibbsianness, and a couple of pioneer numerical studies yielded findings respectively consistent with Gibbsianness [**21**] and non-Gibbsianness [**22**]. However, we hope that the simple non-Gibbsianness mechanism clearly illustrated by our examples could be a useful guide and reference for the study of the more involved two-way-noise situation.

In our examples, non-Gibbsianness shows up in the same way as in the basic voter model [**16**]: Large droplets of aligned ("unanimous") spins have too large probability for the invariant measures to be Gibbsian. More precisely, we show that once a suitable "spider" of "1" appears, the dynamics causes the alignment of the spins in a neighboring sphere. This sort of damage-spreading property (or error-correcting deficiency) implies that the presence of a sphere of "1" is penalized by the invariant measures only as a sub-volume exponential. This contradicts well known Gibbsian properties. In fact, we can be more precise. Gibbsian measures are characterized by two properties [**13**]: uniform non-nullness and quasilocality. As we comment in Section 3, the large probability of aligned droplets means that the invariant measures can not be uniformly non-null. More generally, such invariant measures can not be the result of block renormalizations of non-null, in particular Gibbsian, measures. Furthermore, known arguments [**7**] (briefly reviewed in Section 3 below), imply that if one of these measures is not a product measure, then its non-Gibbsianness is preserved by further single-site renormalization transformations.

2. Simple examples

Before plunging into the technical and notational details needed to describe our results in full generality, we would like to present some simple examples that contain the essential ideas. The examples are defined on the configuration space $\{0,1\}^{\mathbb{Z}^2}$.

Example 1: The NEC model. — Its deterministic version is defined by a translation-invariant parallel updating defined by the rule

$$x_{\mathrm{det}}^{t+1}(0,0) = \mathit{major}\Big\{x^t(0,1)\,,\, x^t(1,0)\,,\, x^t(0,0)\Big\}, \tag{1}$$

where $x^t(i,j)$ denotes the configuration at site $(i,j) \in \mathbb{Z}^2$ immediately after the t-th iteration of the transformation and $\mathit{major} : \{0,1\}^{2k+1} \to \{0,1\}$ is the majority function, i.e. the Boolean function of any odd number of arguments, which equals "1" if and only if most of its arguments equal "1". This prescription yields an evolution, which is symmetric with respect to the flip $0 \leftrightarrow 1$ [a function with this property is called a *self-spin-flip function* in Section 4 below]. We consider a noisy version, where in addition spins "0" flip into "1" independently with a certain probability ε, while spins "1" remain unaltered. This corresponds to stochastic updating

$$\mathrm{Prob}\Big(x^{t+1}(i,j) = 0 \;\Big|\; x^t\Big) = (1-\varepsilon)\,\Big[1 - x_{\mathrm{det}}^{t+1}(i,j)\Big]. \tag{2}$$

The "all-ones" delta-measure δ_1 is invariant for this automaton. For small ε there is at least another invariant measure, as a consequence of Theorem 4.2 below.

Let us start with the following simple observations which are immediate consequences of the NEC rule (1) and the one-sidedness of the noise:

(i) Horizontal lines (parallel to axis i) filled with spins "1" remain invariant under the evolution.

(ii) The same invariance holds for vertical lines (parallel to axis j) filled with spins "1".

(iii) After one evolution-step (that is, after one parallel updating of all the spins), a line of slope -1 filled with spins "1" moves into the parallel line immediately to the south-west.

(iv) If the (infinite) "spider" formed by the i-axis, the j-axis and the line $i+j=0$ is filled with "1", then after t steps the evolution causes the whole triangle $\{(i,j) : i,j \leqslant 0,\ i+j \geqslant -t\}$ to be filled with "1".

The last observation can be visualized as a displacement, at speed 1, of the "front" formed by the line $i+j=0$, with a simultaneous displacement (here a trivial one), at speed 0, of the "fronts" formed by the i- and j-axis. This combined displacement produces a growing triangle full of "1".

The same observations hold if full lines are replaced by finite segments, except that, depending on the values of neighboring spins, in each iteration each segment can lose one or both of the "1" at its endpoints. We conclude that if at some time the spider

$$\begin{aligned}\mathrm{SP}_{(0,0),L} = \Big\{(i,0) \in \mathbb{Z}^2 : -8L \leqslant i \leqslant 4L\Big\} \bigcup \Big\{(0,j) \in \mathbb{Z}^2 : -8L \leqslant j \leqslant 4L\Big\}\\ \bigcup \Big\{(i,j) \in \mathbb{Z}^2 : i+j = 0\,,\ -6L \leqslant i \leqslant 6L\Big\}\end{aligned} \tag{3}$$

is filled with "1", then after $4L$ iterations the "1" fill a triangular region that contains the sphere of radius L centered at $(-L, L)$, to be denoted $S_{(-L,-L),L}$. Therefore, if μ

is a invariant measure,

$$\mu(1_{S_{(-L,-L),L}}) \geqslant \mu(1_{\mathrm{SP}_{(0,0),L}}) \geqslant \varepsilon^{3(12L+1)}. \tag{4}$$

We have denoted 1_Λ, for $\Lambda \subset \mathbb{Z}^2$, the event $\{x : x(i,j) = 1, (i,j) \in \Lambda\}$. The last inequality in (4) follows from the fact that a "1" has a probability at least ε to appear at a given site because of the noise. As commented in Section 3, such a probability is too large for the invariant measure to be Gibbsian, or block-transformed Gibbsian.

Example 2: North-South maximum of minima (NSMM). — The initial deterministic prescription is defined by

$$x^{t+1}_{\mathrm{det}}(0,0) = \max\Big\{\min\Big(x^t(0,0)\,,\, x^t(1,0)\Big)\,,\, \min\Big(x^t(0,1)\,,\, x^t(1,1)\Big)\Big\} \tag{5}$$

plus translation-invariance. The corresponding evolution is not symmetric under flipping, unlike the previous example. The stochastic version is obtained by adding one-sided noise as in (2). For small ε this automaton has more than one invariant measure (see comment after Theorem 4.2). One of them is, of course, the "all-ones" delta-measure δ_1.

The mechanism for non-Gibbsianness for this model is even simpler to describe than for the NEC model. Indeed, it suffices to observe that whenever a horizontal line is filled with "1", then in the next iteration these "1" survive and in addition the parallel line immediately to the South becomes also filled with "1". The same phenomenon happens for finite horizontal segments, except that each creation of a new segment filled with "1" can be accompanied by shrinkages of up to two sites (the spins at the endpoints) of all the previously created segments. We conclude that if the "spider" (which looks more like a snake in this case)

$$\widetilde{\mathrm{SP}}_{(0,0),L} = \Big\{(i,0) \in \mathbb{Z}^2 : -3L \leqslant i \leqslant 3L\Big\} \tag{6}$$

is filled with "1" at some instant, then $2L$ instants later the "1" will cover at least a square region that includes the sphere $S_{(0,-L),L}$. Arguing as for (4), we obtain for all invariant measures μ the bound

$$\mu(1_{S_{(0,-L),L}}) \geqslant \mu(1_{\widetilde{\mathrm{SP}}_{(0,0),L}}) \geqslant \varepsilon^{6L+1}, \tag{7}$$

which implies that μ is neither Gibbsian nor block-transformed Gibbsian.

A comment by A. van Enter (private communication) gives a colorful description of the mechanism acting in both preceding examples: "the spider fills his stomach faster ($\asymp L$ sites at a time) than his legs shrink ($\asymp 1$ sites at a time)".

Example 3: A non-example. — The automata defined by the deterministic prescription

$$\begin{aligned} x^{t+1}_{\mathrm{det}}(0,0) = \mathit{major}\Big\{&\min\Big(x^t(0,2),\, x^t(-1,2)\Big),\, \min\Big(x^t(2,0),\, x^t(2,-1)\Big), \\ &\min\Big(x^t(0,-1),\, x^t(-1,0)\Big)\Big\} \end{aligned} \tag{8}$$