Astérisque

CARLOS GUTIERREZ
ALBERTO SARMIENTO

Injectivity of C^1 Maps $\mathbb{R}^2 \to \mathbb{R}^2$ at infinity and planar vector fields

Astérisque, tome 287 (2003), p. 89-102
<http://www.numdam.org/item?id=AST_2003__287__89_0>

Astérisque
287, 2003, p. 89–102

INJECTIVITY OF C^1 MAPS $\mathbb{R}^2 \to \mathbb{R}^2$ AT INFINITY AND PLANAR VECTOR FIELDS

by

Carlos Gutierrez & Alberto Sarmiento

Abstract. — Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map, where $\sigma > 0$ and $\overline{D}_\sigma = \{p \in \mathbb{R}^2 : ||p|| \leqslant \sigma\}$.
(i) If for some $\varepsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, no eigenvalue of $DX(p)$ belongs to $(-\varepsilon, \infty)$, there exists $s \geqslant \sigma$, such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;
(ii) If for some $\varepsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, no eigenvalue of $DX(p)$ belongs to $(-\varepsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geqslant 0\}$, there exists $p_0 \in \mathbb{R}^2$ such that the point ∞, of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$, is either an attractor or a repellor of $x' = X(x) + p_0$.

1. Introduction

The study of planar vector fields around singularities has somehow motivated the present work. A sample of this study is the work done by C. Chicone, F. Dumortier, J. Sotomayor, R. Roussarie, F. Takens. See for instance [**Chi, DRS, Rou, Tak**]. Here we study the behavior of a vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ around infinity. While a C^1 vector field around a singularity is quite regular, we work under conditions that do not imply, a priori, any regularity of the vector field around infinity. Given an open subset U of $\mathbb{R}^2$ and a C^1 map $Y : U \to \mathbb{R}^2$, we shall denote by $\operatorname{Spec}(Y) = \{\text{eigenvalues of } DY(p) : p \in U\}$. Our main result is the following

Theorem 1. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map, where $\sigma > 0$ and $\overline{D}_\sigma = \{p \in \mathbb{R}^2 : ||p|| \leqslant \sigma\}$. The following is satisfied:*

(i) *if for some $\varepsilon > 0$, $\operatorname{Spec}(X)$ is disjoint of $(-\varepsilon, \infty)$, then there exists $s \geqslant \sigma$, such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;*

2000 ***Mathematics Subject Classification.*** — 37E35, 37C10.
Key words and phrases. — Injectivity, Reeb component, vector fields.

C.G.: Supported in part by PRONEX/FINEP/MCT - grant number 76.97.1080.00.

(ii) *if for some* $\varepsilon > 0$, $\operatorname{Spec}(X)$ *is disjoint of* $(-\varepsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geqslant 0\}$, *then, there exists* $p_0 \in \mathbb{R}^2$ *such that the point* ∞, *of the Riemann sphere* $\mathbb{R}^2 \cup \{\infty\}$,*is either an attractor or a repellor of* $x' = X(x) + p_0$.

To give an idea of the proof of this result, let us introduce the following definition.

Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Since $f : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}$ is a C^1 submersion, $q \in \mathbb{R}^2 \to \nabla f^\#(q) = (-f_y(q), f_x(q))$, the Hamiltonian vector field of f, has no singularities. Let $g_0(x, y) = xy$ and consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] : 0 < x + y \leqslant 2\}.$$

We will say that $\mathcal{A} \subset \mathbb{R}^2$ is a *HRC* (*Half-Reeb Component*) of $\nabla f^\#$ (see figure 1) if there is a homeomorphism $h : B \to \mathcal{A}$ which is a topological equivalence between $\nabla f^\#|_{\mathcal{A}}$ and $\nabla {g_0}^\#|_B$, and such that

(1) $h(\{(x, y) \in B : x + y = 2\})$ (called *the compact edge of* $\mathcal{A}$) is a smooth segment transversal to $\nabla f^\#$ in the complement of $h(1, 1)$, and

(2) both $h(\{(x, y) \in B : x = 0\})$ and $h(\{(x, y) \in B : y = 0\})$ are full half–trajectories of $\nabla f^\#$.

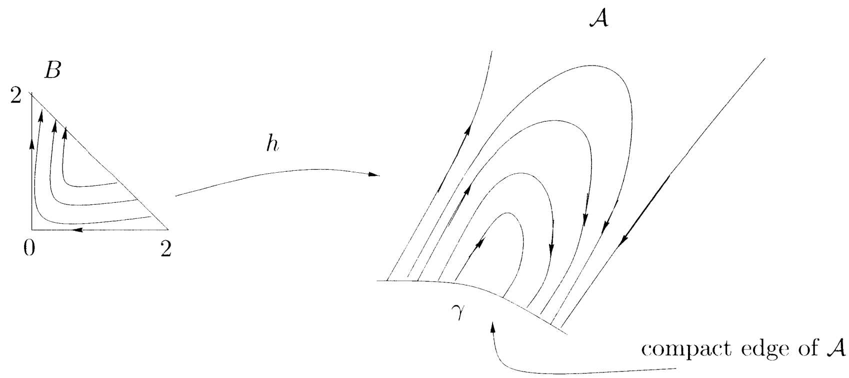

FIGURE 1. A half-Reeb component.

Observe that $\mathcal{A}$ may not be a closed subset of $\mathbb{R}^2$.

Proceed to give an idea of the proof of Theorem 1. First, we shall prove that:

Proposition 1. — *if* $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ *is a* C^1 *map as in Theorem 1, then any HRC of* $\nabla^\# f$ *is a bounded subset of* $\mathbb{R}^2$.

This is used to prove

Theorem 2. — *if* $Y = (\widetilde{f}, \widetilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$ *is a* C^1 *map such that, for some* $\varepsilon > 0$, $\operatorname{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \varnothing$, *then* Y *is injective.*

Roughly speaking about Theorem 2, if the foliation induced by $\nabla \widetilde{f}^{\#}$ has no half-Reeb components then, $\nabla \widetilde{f}^{\#}$ is topologically equivalent to the foliation, on the (x, y)-plane, induced by the form dx (the foliation is made up by all the vertical straight lines). The injectivity of X will follow from the fact that $\nabla \widetilde{f}^{\#}$ and $\nabla \widetilde{g}^{\#}$ are linearly independent everywhere.

Sections 3 and 4 are devoted to prove

Corollary 2. — *if $X = (f, g) : \mathbb{R}^2 \smallsetminus \overline{D}_\sigma \to \mathbb{R}^2$ is a C^1 map as in Theorem 1, then there exists a smooth compact disc E such that $\nabla f^{\#}$, restricted to $\mathbb{R}^2 \smallsetminus E$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \smallsetminus \overline{D}_1$, induced by dx.*

Observe that the foliation, on $\mathbb{R}^2 \smallsetminus \overline{D}_1$, induced by dx has exactly two tangencies with $\partial \overline{D}_1$ (at $(1, 0)$ and $(0, 1)$) which are "quadratic" and "external". Let us say a little more about what is proved in Section 3 and 4: We show, in Section 3, that given any generic smooth compact disc $F \supset \overline{D}_\sigma$ the number of "external" tangencies of ∇f with ∂F is equal to 2 plus the number of "internal" tangencies of ∇f with ∂F. We show, in Section 4, that the disc F can be deformed to a smooth compact disc E so that the referred "external" and "internal" tangencies cancel in pairs yielding exactly 2 tangencies which are "external".

Using Theorem 2 we obtain

Proposition 2. — *Let X be as in Corollary 2. If X takes ∂E diffeomorphically to a circle then $X|_{\mathbb{R}^2 \smallsetminus E}$ may be extended to a map which satisfies conditions of Theorem 2 and so it is injective.*

The proof of item (ii) of Theorem 1 is finished in Sections 5 and 6 by showing that, under conditions of Corollary 2, the disc E can be deformed so that, for the resulting new disc, still denoted by E, $\nabla f^{\#}|_{\mathbb{R}^2 \smallsetminus E}$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \smallsetminus \overline{D}_1$, induced by dx and moreover X takes ∂E diffeomorphically to a circle. Then the result follows from Proposition 2.

The item (ii) of Theorem 1 follows from the corresponding item (i) and some previous Gutierrez and Teixeira work [**G-T**].

Throughout this article, given an embedded circle $C \subset \mathbb{R}^2$, the compact disc (resp. open disc) bounded by C will be denoted by $\overline{D}(C)$ (resp. $D(C)$). Also, we will freely use the fact that the assumptions of the theorem are open in the Whitney C^1–topology. In this way, when possible and necessary, we will assume that X is smooth and that it satisfies some generic property which will be made precise at the proper place.

Acknowledgements. — We wish to thank referee's comments which have been appreciated and incorporated into this work.

2. A global injectivity result

We shall need the following lemma which is contained in the proof of [**Gut**, Lemma 2.5]. For $\theta \in \mathbb{R}$: let R_θ denote the linear rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Lemma 1. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Suppose that $\nabla^\# f$ has an HRC which is unbounded (as a subset of $\mathbb{R}^2$) but whose projection on the x-axis is a compact interval. Then, there exists $\varepsilon > 0$ such that, for all $\theta \in (-\varepsilon, 0) \cup (0, \varepsilon)$ $\nabla^\# f_\theta$ has a HRC whose projection on the x-axis is an interval of infinite length; here $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$.*

The proof of Proposition 1 and Theorem 2 can be found in [**CGL**] but, as we have already said and for sake of completeness, they are included here.

Proposition 1. — *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Then any HRC of $\nabla^\# f$ is a bounded subset of $\mathbb{R}^2$.*

Proof. — Let $\mathcal{A}$ be a half Reeb component for f. Let $\Pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate. By composing with a rotation if necessary, in the way that is stated in Lemma 1, we may suppose that $\Pi(\mathcal{A})$ is an interval of infinite length, say $[b, \infty)$. We may also assume that X is smooth and —by Thom's Transversality Theorem for jets [**G-G**]— that

(a1) the set

$$T = \{(x, y) \in \mathbb{R}^2 : f_y(x, y) = 0\}$$

is made up of regular curves;

(a2) There is a discrete subset Δ of T such that if $p \in T \setminus \Delta$ (resp. $p \in \Delta$), $\nabla^\# f$ has quadratic contact (resp. cubic contact) with the vertical foliation of $\mathbb{R}^2$.

Then, if $a > b$ is large enough,

(b) for any $x \geqslant a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\nabla f^\#|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap (x, \infty) = \varnothing$; in other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

It follows that

(c) if $x \geqslant a$ and $p \in \alpha_x \cap \Pi^{-1}(x)$ then $p \in T \cap \mathcal{A} \setminus \Delta$.

Let T_m be the set of $p \in \mathcal{A}$ such that, for some $x \geqslant a$, $p \in \alpha_x \cap \Pi^{-1}(x)$. Notice that, for every $x \geqslant a$, $\alpha_x \cap \Pi^{-1}(x)$ is a finite set; nevertheless, by (b), (c) and by using Thom's Transversality Theorem for jets, we may get the following stronger statement:

(d) There is a sequence $F = \{a_1, a_2, \ldots, a_i, \cdots\}$ in $[a, \infty)$, which may be either empty or finite or else countable, such that if $x \in F$ (resp. $x \in [a, \infty) \setminus F$), then $\Pi^{-1}(x) \cap T_m$ is a two-point-set (resp. a one-point-set).