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Yves Laurent

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# $b$-FUNCTIONS AND INTEGRABLE SOLUTIONS OF HOLONOMIC $\mathcal{D}$-MODULE

*by*

Yves Laurent

*À Jean-Pierre Ramis, à l'occasion de son 60$^e$ anniversaire.*

***Abstract.*** — A famous theorem of Harish-Chandra shows that all invariant eigendistributions on a semi-simple Lie group are locally integrable functions. We give here an algebraic version of this theorem in terms of polynomials associated with a holonomic $\mathcal{D}$-module.

***Résumé*** **($b$-fonctions et solutions intégrables des modules holonomes).** — Un célèbre théorème de Harish-Chandra montre que les distributions invariantes propres sur un groupe de Lie semi-simple sont des fonctions localement intégrables. Nous donnons ici une version algébrique de ce théorème en termes de polynômes associés à un $\mathcal{D}$-module holonome.

## Introduction

Let $G_{\mathbb{R}}$ be a real semisimple Lie group and $\mathfrak{g}_{\mathbb{R}}$ be its Lie algebra. An *invariant eigendistribution* $T$ on $G_{\mathbb{R}}$ is a distribution which is invariant under conjugation by elements of $G_{\mathbb{R}}$ and is an eigenvector of every bi-invariant differential operator on $G_{\mathbb{R}}$. The main examples of such distributions are the characters of irreducible representations of $G_{\mathbb{R}}$. A famous theorem of Harish-Chandra sets that all invariant eigendistributions are $L^1_{\text{loc}}$-functions on $G_{\mathbb{R}}$ [**4**]. After transfer to the Lie algebra by the exponential map, such a distribution satisfies a system of partial differential equations.

In the language of $\mathcal{D}$-modules, these equations define a holonomic $\mathcal{D}$-module on the complexified Lie algebra $\mathfrak{g}$. We call this module the Hotta-Kashiwara module as it has been defined and studied first in [**6**]. In [**20**], J. Sekiguchi extended these results to symmetric pairs. He proved in particular that a condition on the symmetric pair is needed to extend Harish-Chandra theorem. In several papers, Levasseur and Stafford [**15, 16, 17**] gave an algebraic proof of the main part of Harish-Chandra theorem.

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In [**3**], we defined a class of holonomic $\mathcal{D}$-modules. which we called *tame $\mathcal{D}$-modules*. These $\mathcal{D}$-modules have no quotients supported by a hypersurface and their distribution solution are locally integrable. We proved in particular that the Hotta-Kashiwara module is tame, recovering Harish-Chandra theorem. The definition of tame is a condition on the roots of the $b$-functions which are polynomials attached to the $\mathcal{D}$-module and a stratification of the base space. However, the proof of the fact that the Hotta-Kashiwara module is tame involved some non algebraic vector fields.

The first aim of this paper is to give a completely algebraic version of Harish-Chandra theorem. We give a slightly different definition of tame and an algebraic proof of the fact that the Hotta-Kashiwara module is tame. This proof is different from the proof of [**3**] and gives more precise results on the roots of the $b$-functions. However our first proof was still valid in the case of symmetric pairs while the present proof uses a morphism of Harish-Chandra which does not exist in that case.

Our second aim is to answer to a remark made by Varadarajan during the Ramis congress. He pointed the fact that an invariant eigendistribution, considered as a distribution on the Lie algebra by the exponential map, is not a solution of the Hotta-Kashiwara module. A key point in the original proof of Harish-Chandra is precisely the proof that after multiplication by a function, the eigendistribution is solution of the Hotta-Kashiwara module (see [**23**]). The study of the Hotta-Kashiwara module did not bypass this difficult step. Here we consider a family of holonomic $\mathcal{D}$-module, which we call (H-C)-modules; this family includes the Hotta-Kashiwara modules but also the module satisfied directly by an eigendistribution. We prove that these modules are tame and get a direct proof of Harish-Chandra theorem.

## 1. $V$-filtration and $b$-functions

We first recall the definition and a few properties of the classical $V$-filtration, then we give a new definition of quasi-homogeneous $b$-functions and of tame $\mathcal{D}$-modules. We end this section with a result on the inverse image of $\mathcal{D}$-modules which will be a key point of the proof in the next section.

**1.1. Standard $V$-filtrations.** — In this paper, $(X, \mathcal{O}_X)$ is a smooth algebraic variety defined over $k$, an algebraically closed field of characteristic 0. The sheaf of differential operators with coefficients in $\mathcal{O}_X$ is denoted by $\mathcal{D}_X$. Results and proofs are still valid if $k = \mathbb{C}$, $X$ is a complex analytic manifold and $\mathcal{D}_X$ is the sheaf of differential operators with holomorphic coefficients.

Let $Y$ be a smooth subvariety of $X$ and $\mathcal{I}_Y$ the ideal of definition of $Y$. The $V$-filtration along $Y$ is given by [**10**]:

$$V_k\mathcal{D}_X = \{\, P \in \mathcal{D}_X|_Y \mid \forall\, l \in \mathbb{Z}, P\mathcal{I}_Y^l \subset \mathcal{I}_Y^{l+k} \,\}$$

(with $\mathcal{I}_Y^l = \mathcal{O}_X$ if $l \leqslant 0$).

This filtration has been widely used in the theory of $\mathcal{D}$-modules, let us recall some of its properties (for the details, we refer to [**19**], [**12**], [**18**], [**14**]). The associated graded ring $\mathrm{gr}_V \mathcal{D}_X$ is the direct image by $p : T_Y X \to X$ of the sheaf $\mathcal{D}_{T_Y X}$ of differential operators on the normal bundle $T_Y X$. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, a $V\mathcal{D}_X$-filtration on $\mathcal{M}$ is a good filtration if it is locally finite, *i.e.* if, locally, there are sections $(u_1, \ldots, u_N)$ of $\mathcal{M}$ and integers $(k_1, \ldots, k_N)$ such that $V_k\mathcal{M} = \sum V_{k-k_i}\mathcal{D}_X u_i$.

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module provided with a good $V$-filtration, the associated graded module is a coherent $\mathrm{gr}_V \mathcal{D}_X$-module and if $\mathcal{N}$ is a coherent submodule of $\mathcal{M}$ the induced filtration is a good filtration (see [**19**, Chapter III, Proposition 1.4.3] or [**18**]).

Let $\theta_Y$ be the Euler vector field of the fiber bundle $T_Y X$, that is the vector field verifying $\theta_Y(f) = kf$ when $f$ is a function on $T_Y X$ homogeneous of degree $k$ in the fibers of $p$. A *b-function* along $Y$ for a coherent $\mathcal{D}_X$-module with a good $V$-filtration is a polynomial $b$ such that

$$\forall k \in \mathbb{Z}, \qquad b(\theta_Y + k)\mathrm{gr}_V^k \mathcal{M} = 0$$

If the good $V$-filtration is replaced by another, the roots of $b$ are translated by integers. Here, we always fix the filtration, in particular, if the $\mathcal{D}_X$-module is of the type $\mathcal{D}_X/\mathcal{I}$, the good filtration will be induced by the canonical filtration of $\mathcal{D}_X$.

**1.2. Quasi-homogeneous $V$-filtrations and quasi-$b$-functions.** — Let $\varphi = (\varphi_1, \ldots, \varphi_d)$ be a polynomial map from $X$ to the vector space $W = k^d$ and $m_1, \ldots, m_d$ be strictly positive and relatively prime integers. We define a filtration on $\mathcal{O}_X$ by:

$$V_k^\varphi \mathcal{O}_X = \sum_{\langle m,\alpha\rangle = -k} \mathcal{O}_X \varphi^\alpha$$

with $\alpha \in \mathbb{N}^d$, $\langle m, \alpha\rangle = \sum m_i\alpha_i$ and $\varphi^\alpha = \varphi_1^{\alpha_1}\cdots\varphi_d^{\alpha_d}$. If $k \geqslant 0$ we set $V_k^\varphi\mathcal{O}_X = \mathcal{O}_X$.

This filtration extends to $\mathcal{D}_X$ by:

$$(1) \qquad V_k^\varphi \mathcal{D}_X = \{ P \in \mathcal{D}_X \mid \forall l \in \mathbb{Z}, PV_l^\varphi\mathcal{O}_X \subset V_{l+k}^\varphi \mathcal{O}_X \}$$

***Definition 1.2.1.*** A $(\varphi, m)$-weighted Euler vector field is a vector field $\eta$ in $\sum_i \varphi_i \mathcal{V}_X$ such that $\eta(\varphi_i) = m_i\varphi_i$ for $i = 1, \ldots, d$. ($\mathcal{V}_X$ is the sheaf of vector fields on $X$.)

***Lemma 1.2.2.*** *Any $(\varphi, m)$-weighted Euler vector field is in $V_0^\varphi\mathcal{D}_X$ and if $\eta_1$ and $\eta_2$ are two $(\varphi, m)$-weighted Euler vector fields, $\eta_1 - \eta_2$ is in $V_{-1}^\varphi\mathcal{D}_X$.*

The map $\varphi$ may be not defined on $X$ but on an étale covering of $X$. More precisely, let us consider an étale morphism $\nu : X' \to X$ and a morphism $\varphi : X' \to W = k^d$. If $m_1, \ldots, m_d$ are strictly positive and relatively prime integers, we define $V_k^\varphi\mathcal{O}_X$ as the sheaf of functions on $X$ such that $f\circ\nu$ is in $V_k^\varphi\mathcal{O}_{X'}$. This defines a $V$-filtration on $\mathcal{O}_X$ and on $\mathcal{D}_X$ by the formula (1). The map $TX' \to TX \times_X X'$ is an isomorphism and a vector field $\eta$ on $X$ defines a unique vector field $\nu^*(\eta)$ on $X'$. By definition, a vector field $\eta$ on $X$ is a $(\varphi, m)$-weighted Euler vector field if $\nu^*(\eta)$ is a $(\varphi, m)$-weighted Euler vector field on $X'$.

***Definition 1.2.3.*** — Let $u$ be a section of a coherent $\mathcal{D}_X$-module $\mathcal{M}$. A polynomial $b$ is a quasi-$b$-function of type $(\varphi, m)$ for $u$ if there exist a $(\varphi, m)$-weighted Euler vector field $\eta$ and a differential operator $Q$ in $V_{-1}^{\varphi}\mathcal{D}_X$ such that $(b(\eta) + Q)u = 0$.

The quasi-$b$-function is said *regular* if the order of $Q$ as a differential operator is less or equal to the order of the polynomial $b$ and *monodromic* if $Q = 0$.

The quasi-$b$-function is said *tame* if the roots of $b$ are strictly greater than $-\sum m_i$.

These definitions are valid for any map $\varphi$ but here we always assume that $\varphi$ is smooth. Then if $Y = \varphi^{-1}(0)$, we say for short that $b$ is a quasi-$b$-function of total weight $|m| = \sum m_i$ along $Y$. Remark that lemma 1.2.2 shows that the definition is independent of the $(\varphi, m)$-weighted Euler vector field $\eta$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. A $V^{\varphi}\mathcal{D}_X$-filtration on $\mathcal{M}$ is a good filtration if it is locally finite.

***Definition 1.2.4.*** — Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $V^{\varphi}\mathcal{M}$ a good $V^{\varphi}\mathcal{D}_X$-filtration. A polynomial $b$ is a quasi-$b$-function of type $(\varphi, m)$ for $V^{\varphi}\mathcal{M}$ if, for any $k \in \mathbb{Z}$, $b(\eta + k)V_k^{\varphi}\mathcal{M} \subset V_{k-1}^{\varphi}\mathcal{M}$ where $\eta$ is a $(\varphi, m)$-weighted Euler vector field.

The quasi-$b$-function is *monodromic* if $b(\eta + k)V_k^{\varphi}\mathcal{M} = 0$.

Definition 1.2.3 is a special case of definition 1.2.4 if $\mathcal{D}_X u$ is provided with the filtration induced by the canonical filtration of $\mathcal{D}_X$.

Recall that if $\mathcal{M}$ is a $\mathcal{D}_X$-module its inverse image by $\nu$ is its inverse image as an $\mathcal{O}_X$-module, that is:

$$\nu^+\mathcal{M} = \mathcal{O}_{X'} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\mathcal{M} = \mathcal{D}_{X'\to X} \otimes_{\nu^{-1}\mathcal{D}_X} \nu^{-1}\mathcal{M}$$

where $\mathcal{D}_{X'\to X}$ is the $(\mathcal{D}_{X'}, \nu^{-1}\mathcal{D}_X)$-bimodule $\mathcal{O}_{X'} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\mathcal{D}_X$.

***Lemma 1.2.5.*** — *Let $\nu : X' \to X$ be an étale morphism and let $\varphi$ be a morphism $X' \to W = k^d$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

*The polynomial $b$ is a quasi-b-function of type $(\varphi, m)$ for a section $u$ of $\mathcal{M}$ if and only if it is a quasi-b-function of type $(\varphi, m)$ for the section $1 \otimes u$ of $\nu^+\mathcal{M}$.*

*Proof.* — If $\nu : X' \to X$ is étale, the canonical morphism $\mathcal{D}_{X'} \to \mathcal{D}_{X'\to X}$ given by $P \mapsto P(1 \otimes 1)$ is an isomorphism and defines an injective morphism $\nu^* : \nu^{-1}\mathcal{D}_X \to \mathcal{D}_{X'}$.

Conversely, the morphism $\tilde{\nu} : \nu_*\mathcal{O}_{X'} \to \mathcal{O}_X$ given by $\tilde{\nu}(f)(x) = \sum_{y\in\nu^{-1}(x)} f(y)$ extends to a morphism $\nu_*\mathcal{D}_{X'} \to \mathcal{D}_X$.

These two morphism are compatible with the $V$-filtration defined by $\varphi$ and, by definition, a vector field $\eta$ on $X$ is a $(\varphi, m)$-weighted Euler vector field if and only if $\nu^*(\eta)$ is a $(\varphi, m)$-weighted Euler vector field on $X'$. If $(b(\eta) + R)u = 0$ we