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CURVATURE OF PENCILS OF FOLIATIONS

by

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Dedicated to J.-P. Ramis in his 60^{th} birthday

Abstract. — Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. In this case, we can define the *pencil* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ *of foliations generated by* $\mathcal{F}$ *and* $\mathcal{G}$. We can associate to a pencil $\mathcal{P}$ a meromorphic 2-form $\Theta = \Theta(\mathcal{P})$, the form of curvature of the pencil, which is in fact the Chern curvature (*cf.* [**Ch**]). When $\Theta(\mathcal{P}) \equiv 0$ we will say that the pencil is *flat*. In this paper we give some sufficient conditions for a pencil to be flat. (Theorem 2). We will see also how the flatness reflects in the pseudo-group of holonomy of the foliations of $\mathcal{P}$. In particular, we will study the set $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H} \text{ has a first integral}\}$ in some cases (Theorem 1).

***Résumé* (Courbure de pinceaux de feuilletages).** — Nous nous intéressons au *pinceau de feuilletages* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ engendré par deux feuilletages $\mathcal{F}$ et $\mathcal{G}$ holomorphes singuliers distincts sur une surface complexe compacte M et appartenant à la même classe, *i.e.*, $N_{\mathcal{F}} = N_{\mathcal{G}}$. La forme de courbure du pinceau $\mathcal{P}$ est une 2-forme $\Theta = \Theta(\mathcal{P})$ qui coïncide avec la courbure de Chern (*cf.* [**Ch**]) ; lorsque $\Theta(\mathcal{P}) \equiv 0$ on dit que le pinceau est *plat*. Dans cet article, nous donnons des conditions suffisantes de platitude d'un pinceau (Théorème 2). Nous regardons comment se traduit la platitude dans le pseudo-groupe d'holonomie des feuilletages de $\mathcal{P}$ et, en particulier, nous étudions dans certains cas l'ensemble $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H} \text{ admet une intégrale première}\}$ (Théorème 1).

1. Introduction

Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, with isolated singularities, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. This means that there exists a Leray covering $(U_\alpha)_{\alpha \in A}$ of M by open sets, and collections $(\omega_\alpha)_{\alpha \in A}$, $(\eta_\alpha)_{\alpha \in A}$ and $(g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, such that

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(I) ω_α and η_α are holomorphic 1-forms on U_α which represent the foliations $\mathcal{F}$ and $\mathcal{G}$, respectively. This means that $\mathcal{F}|_{U_\alpha}$ and $\mathcal{G}|_{U_\alpha}$ are defined by the differential equations $\omega_\alpha = 0$ and $\eta_\alpha = 0$, respectively. Since the singularities of $\mathcal{F}$ and $\mathcal{G}$ are isolated, we have $\mathrm{cod}_{\mathbb{C}}(\omega_\alpha = 0) \geqslant 2$ and $\mathrm{cod}_{\mathbb{C}}(\eta_\alpha = 0) \geqslant 2$ for every $\alpha \in A$.

(II) If $U_{\alpha\beta} \neq \varnothing$ then $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$, $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$ and $\eta_\alpha = g_{\alpha\beta} \cdot \eta_\beta$ on $U_{\alpha\beta}$.

The class of the multiplicative cocycle $(g_{\alpha\beta})_{U_{\alpha\beta}\neq\varnothing}$ in $\mathrm{Pic}(M)$ defines $N_{\mathcal{F}}$ and $N_{\mathcal{G}}$, so that $N_{\mathcal{F}} = N_{\mathcal{G}}$. The *pencil generated by $\mathcal{F}$ and $\mathcal{G}$* is the family $\mathcal{P} = (\mathcal{F}_T)_{T\in\overline{\mathbb{C}}}$, where

(III) $\mathcal{F}_\infty = \mathcal{G}$ and if $T \in \mathbb{C}$, then $\mathcal{F}_T$ is represented on U_α by the form $\omega_\alpha^T := \omega_\alpha + T \cdot \eta_\alpha$.

The singular set of $\mathcal{F}_T$ is defined by $\mathrm{sing}(\mathcal{F}_T) \cap U_\alpha = \{\omega_\alpha^T = 0\}$. The tangency divisor of $\mathcal{F}$ and $\mathcal{G}$ is defined by $\mathrm{Tang}(\mathcal{F},\mathcal{G}) \cap U_\alpha = \{\omega_\alpha \wedge \eta_\alpha = 0\}$. Note that $\mathrm{sing}(\mathcal{F}_T)$ and $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ are analytic subsets of M and that $\mathrm{sing}(\mathcal{F}_T) \subset |\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ for all $T \in \overline{\mathbb{C}}$. Since $\mathcal{F} \neq \mathcal{G}$, $|\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ is a proper analytic subset of pure dimension one. Let $W = M \setminus |\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ and $W_\alpha = W \cap U_\alpha$. Since $\omega_\alpha \wedge \eta_\alpha(p) \neq 0$ for all $p \in W_\alpha$, there exists an unique holomorphic 1-form θ_α on W_α such that

$$(*)\qquad d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha \quad \text{and} \quad d\eta_\alpha = \theta_\alpha \wedge \eta_\alpha$$

for all $\alpha \in A$. It follows from $(*)$, (II) and $\omega_\alpha \wedge \eta_\alpha \not\equiv 0$ that, if $W_{\alpha\beta} := W_\alpha \cap W_\beta \neq \varnothing$ then, $\theta_\alpha = \theta_\beta + dg_{\alpha\beta}/g_{\alpha\beta}$ on $W_{\alpha\beta}$. Hence $d\theta_\alpha = d\theta_\beta$ on $W_{\alpha\beta}$ and we can define a holomorphic 2-form Θ on W by

$$(**)\qquad \Theta|_{U_\alpha} := d\theta_\alpha$$

It can be proved that the form Θ can be extended meromorphically to $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ (see § 2). This extension will be called the *curvature of the pencil $\mathcal{P}(\mathcal{F},\mathcal{G})$*. We will say that the pencil is *flat* if $\Theta = 0$. Let us see some examples of flat pencils.

Example 1. — Let ω and η be two meromorphic closed 1-forms on some compact complex surface M, such that $\omega \wedge \eta \not\equiv 0$ and the divisors of poles and zeroes of ω and η coincide. Let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by ω and η, respectively. It is known that $N_{\mathcal{F}} = N_{\mathcal{G}}$ in this case (*cf.* [**Br**]). Moreover, the pencil generated by $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{P}(\mathcal{F},\mathcal{G})$, is defined by the pencil of forms $\omega_T = \eta + T \cdot \omega$. Therefore, it is flat. We will call a pencil like this *a pencil of closed forms*.

A particular case is given by some families of logarithmic forms in $\mathbb{C}P(2)$. Let $f_1, \ldots, f_k$, $k \geqslant 3$, be irreducible homogeneous polynomials of three variables such that $df_i \wedge df_j \neq 0$ if $i \neq j$. Given $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$, such that $\sum_{j=1}^k \lambda_j \cdot dg(f_j) = 0$, set $\omega_\lambda = \sum_{j=1}^k \lambda_j \cdot df_j/f_j$. The closed form ω_λ can be considered as meromorphic form on $\mathbb{C}P(2)$, so that the family $(\omega_\lambda)_\lambda$ generates a family of foliations $(\mathcal{F}_\lambda)_\lambda$ on $\mathbb{C}P(2)$. It can be checked that any pencil contained in this family is flat.

Another particular case, is the following: let M be the complex two torus $\mathbb{C}^2/\Gamma$, where $\Gamma = \mathbb{Z} \cdot v_1 \oplus \mathbb{Z} \cdot v_2 \oplus \mathbb{Z} \cdot v_3 \oplus \mathbb{Z} \cdot v_4$ is some lattice in $\mathbb{C}^2$, and $\pi \colon \mathbb{C}^2 \to \mathbb{C}^2/\Gamma$ be the canonical projection. Consider an affine coordinate system (z, w) on $\mathbb{C}^2$ and let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by the closed forms ω and η such that $\pi^*(\omega) = dz$ and $\pi^*(\eta) = dw$, respectively.

Example 2. — The pull-back of a flat pencil is a flat pencil. More precisely, let M and N be complex surfaces and $f\colon M - \to N$ be a meromorphic map. If $\mathcal{P} := \mathcal{P}(\mathcal{F}, \mathcal{G})$ is a pencil of foliations on N, then we can define the pencil $f^*(\mathcal{P}) = \mathcal{P}(f^*(\mathcal{F}), f^*(\mathcal{G}))$ on M. It is not difficult to prove that, if $\mathcal{P}$ is flat then $f^*(\mathcal{P})$ is also flat.

Example 3. — Suppose that the pencil $\mathcal{P}(\mathcal{F}, \mathcal{G})$ is defined by $\omega + T \cdot \eta$, where ω and η are meromorphic 1-forms, and there exists a closed meromorphic 1-form θ such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$. Then the pencil $\mathcal{P}(\mathcal{F}, \mathcal{G})$ is flat. Of course, the pencils of Example 1 are of this kind, because the forms ω and η are closed. However, the reader can find some examples in [**LN**] or [**LN-1**] which are not generated by closed forms. One example of this kind is the pencil $\mathcal{P}_1$ of foliations of degree two on $\mathbb{C}P(2)$ defined in some affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ by the the forms (see § 2.4 of [**LN**]):

$$(1) \qquad \begin{cases} \omega_1 = (4x - 9x^2 + y^2)dy - 6y(1 - 2x)dx \\ \eta_1 = 2y(1 - 2x)dy - 3(x^2 - y^2)dx. \end{cases}$$

A straightforward computation gives $d\omega_1 = \frac{5}{6}\frac{dP}{P} \wedge \omega_1$ and $d\eta_1 = \frac{5}{6}\frac{dP}{P} \wedge \eta_1$, where $P(x, y) = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4$. The other examples of [**LN**] can be obtained from the above one by pulling-back $\mathcal{P}_1$ by a meromorphic map $f\colon \mathbb{C}P(2) - \to \mathbb{C}P(2)$.

Another example is the pencil $\mathcal{P}_2$ of degree three generated by

$$(2) \qquad \begin{cases} \omega_2 = y(x^2 - y^2)dy - 2x(y^2 - 1)dx \\ \eta_2 = (4x - x^3 - x^2y - 3xy^2 + y^3)dy + 2(x + y)(y^2 - 1)dx. \end{cases}$$

In this case, we have $d\omega_2 = \frac{3}{4}\frac{dQ}{Q} \wedge \omega_2$ and $d\eta_2 = \frac{3}{4}\frac{dQ}{Q} \wedge \eta_2$, where $Q(x, y) = (y^2 - 1)(x^2 + y^2 - 2x)(x^2 + y^2 + 2x)$.

We would like to observe that both pencils $\mathcal{P}_1$ and $\mathcal{P}_2$ are exceptional families of foliations in the sense of [**LN-1**]. This means the folowing: Let $\mathcal{F}^j_T$, $T \in \overline{\mathbb{C}}$, be the foliation defined in $\mathbb{C}^2 \subset \mathbb{C}P(2)$ by the form $\omega_j + T \cdot \eta_j$ ($\mathcal{F}^j_\infty$ defined by η_j), where ω_j and η_j are as in (j), $j = 1, 2$, of example 3. Then, for $j = 1, 2$, we have:

(a) The singularities of $\mathcal{F}^j_T$ are of constant analytic type. In other words, there is a finite subset $F_j \subset \overline{\mathbb{C}}$ such that if $T_1, T_2 \in \overline{\mathbb{C}} \setminus F_j$ then every singularity of $\mathcal{F}^j_{T_1}$ is locally analytically equivalent to some singularity of $\mathcal{F}^j_{T_2}$.

(b) If we set

$$E_j = \{T \in \overline{\mathbb{C}} \mid \mathcal{F}^j_T \text{ has a meromorphic first integral}\},$$

then E_j is countable and dense in $\overline{\mathbb{C}}$.

(c) Given $T \in E_j$ denote by $d_j(T)$ the degree of the generic level of the first integral of $\mathcal{F}^j_T$. Then, for any $m \in \mathbb{N}$ the set $\{T \in E_j \mid d_j(T) \leqslant m\}$ is finite. In particular, in both families, there are foliations with first integrals of arbitrarily large degrees.

Concerning the exceptional pencils above, we have the following result:

Theorem 1. — *Let* E_j, $j = 1, 2$, *be as in* (b). *Then*

$$\begin{cases} E_1 = \mathbb{Q} \cdot \langle 1, e^{2\pi i/3} \rangle \cup \{\infty\} \\ E_2 = \mathbb{Q} \cdot \langle 1, i \rangle \cup \{\infty\}. \end{cases}$$

where $\mathbb{Q} \cdot \langle a, b \rangle = \{q_1 \cdot a + q_2 \cdot b \mid q_1, q_2 \in \mathbb{Q}\}$.

In our last result we will give some sufficient condictions for the flatness of a pencil $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ in terms of the singularities of the foliations in $\mathcal{P}$ and the components of the divisor of tangencies. In order to state it, let us consider the singularities of $\mathcal{F}_T$, $T \in \overline{\mathbb{C}}$. Without lost of generality, we will suppose that $\mathcal{F}$ and $\mathcal{G}$ have isolated singularities. This implies that the set $NI := \{T \in \overline{\mathbb{C}} \mid \mathcal{F}_T$ has non-isolated singularities$\}$ is finite. Set $IS := \overline{\mathbb{C}} \smallsetminus NI$ and for each $T \in IS$, set $n(T) := \#(\mathrm{sing}(\mathcal{F}_T))$. Note that, if $T \in IS$ then $N_{\mathcal{F}_T} = N_{\mathcal{F}}$. It is well known that the number of singularities of $\mathcal{F}_T$, counted with multiplicities, is given by (*cf.* [**Br**]):

$$m(\mathcal{F}) = m(\mathcal{F}_T) = N_{\mathcal{F}}^2 + N_{\mathcal{F}}.K_M + c_2(M)$$

where K_M is the canonical bundle of M. Hence $n(T) \leqslant m(\mathcal{F})$ for all $T \in IS$. Let $n_0 = \max\{n(T) \mid T \in IS\}$ and $GP = \{T \in IS \mid n(T) = n_0\}$. We need a fact.

Lemma 1. — $\overline{\mathbb{C}} \smallsetminus GP$ *is finite. Moreover, there exist holomorphic maps* $p_j : GP \to M$, $j = 1, \ldots, n_0$, *such that* $\mathrm{sing}(\mathcal{F}_T) = \{p_1(T), \ldots, p_{n_0}(T)\}$ *for all* $T \in GP$.

The proof of Lemma 1 is left for the reader.

Definition 1. — We say that the singularity p_j is *fixed* if the map $p_j : GP \to M$ is constant. Otherwise, we say that p_j is *movable*. For instance, if p is a singularity of the curve $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ then p is a singularity of all foliations of the pencil and it is a fixed singularity of the pencil.

Note that, for any movable singularity p_j of the pencil, the image $p_j(GP)$ is contained in some irreducible component C of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$. In this case we will say that p_j *is contained in* C.

Let $C \subset \mathrm{Tang}(\mathcal{F}, \mathcal{G})$ be an irreducible component. We have two possibilities:

(A) C is invariant for both foliations $\mathcal{F}$ and $\mathcal{G}$. In this case, C is invariant for all foliations $\mathcal{F}_T$ in the pencil and we will say that C is *invariant for the pencil*.

(B) C is not invariant for the pencil. In this case, the set $IN(C) = \{T \in \overline{\mathbb{C}} \mid C$ is invariant for $\mathcal{F}_T\}$ is finite.