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ALGEBRAIC CONSTRUCTION OF THE STOKES SHEAF FOR IRREGULAR LINEAR q-DIFFERENCE EQUATIONS

by

Jacques Sauloy

Je laisse aux nombreux avenirs (non à tous)
mon jardin aux sentiers qui bifurquent.
Jorge Luis Borges, *Fictions*

Abstract. — The local analytic classification of irregular linear q-difference equations has recently been obtained by J.-P. Ramis, J. Sauloy and C. Zhang. Their description involves a q-analog of the Stokes sheaf and theorems of Malgrange-Sibuya type and is based on a discrete summation process due to C. Zhang. We show here another road to some of these results by algebraic means and we describe the q-Gevrey devissage of the q-Stokes sheaf by holomorphic vector bundles over an elliptic curve.

Résumé **(Construction algébrique du faisceau de Stokes pour les équations aux q-différences linéaires irrégulières)**

La classification analytique locale des équations aux q-différences linéaires irrégulières a été récemment réalisée par J.-P. Ramis, J. Sauloy et C. Zhang. Leur description fait intervenir un q-analogue du faisceau de Stokes et des théorèmes de type Malgrange-Sibuya et elle s'appuie sur la sommation discrète de C. Zhang. Nous montrons ici comment retrouver une partie de ces résultats par voie algébrique et nous décrivons le dévissage q-Gevrey du q-faisceau de Stokes par des fibrés vectoriels holomorphes sur une courbe elliptique.

1. Introduction and general conventions

1.1. Introduction. — This paper deals with Birkhoff's program of 1941 ([**3**], see also [**2**]) towards the local analytic classification of q-difference equations and some extensions stated by J.-P. Ramis in 1990 ([**13**]).

A full treatment of the Birkhoff program including the case of irregular q-difference equations is being given in [**16**]. The method used there closely follows the analytic procedure developed in the last decades by B. Malgrange, Y. Sibuya, J.-P. Ramis,... for the "classical" case, *i.e.*, the case of differential equations: adequate asymptotics,

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q-Stokes phenomenon, q-Stokes sheaves and theorems of Malgrange-Sibuya type; explicit cocycles are built using a *discrete summation process* due to C. Zhang ([**27**]) where the Jackson q-integral and theta functions are introduced in place of the Laplace integral and exponential kernels.

To get an idea of the classical theory for linear differential equations one should look at the survey [**25**] by V.S. Varadarajan, especially section 6, and to get some feeling of how the change of landscape from differential equations to q-difference equations operates, at the survey [**7**] by L. Di Vizio, J.-P. Ramis, J. Sauloy and C. Zhang.

The aim of this paper is to show how the harder analytic tools can, to some extent, be replaced by much simpler algebraic arguments. The problem under consideration being a transcendental one we necessarily keep using analytic arguments but in their most basic, "19th century style", features only. In particular, we avoid here using the discrete summation process.

Again our motivation is strongly pushed ahead by the classical model of which we recall three main steps: the *dévissage Gevrey* introduced by J.-P. Ramis ([**12**]) occured to be the fundamental tool for understanding the Stokes phenomenon. The underlying algebra was clarified by P. Deligne in [**4**], then put at work by D.G. Babbitt and V.S. Varadarajan in [**1**] (see also [**25**]) for moduli theoretic purposes. On the same basis, effective methods, a natural summation and galoisian properties were thoroughly explored by M. Loday-Richaud in [**9**].

Some specificities of our problem are due, on one hand, to the fact that the sheaves to be considered are quite similar to holomorphic vector bundles over an elliptic curve, whence the benefit of GAGA theorems, and, on another hand, to the existence for q-difference operators of an analytic factorisation without equivalent for differential operators. Such a factorisation originates in Birkhoff ([**3**]), where it was rather stated in terms of a triangular form of the system. It has been revived by C. Zhang ([**26**], [**10**]) in terms of factorisation and we will use it in its linear guise, as a filtration of q-difference modules ([**22**]).

In this paper, following the classical theory recalled above, we build a q-Gevrey filtration on the q-Stokes sheaf, thereby providing a q-analog of the Gevrey devissage in the classical case. This q-devissage jointly with a natural summation argument allows us to prove the q-analog of a Malgrange-Sibuya theorem (theorem 3 of [**25**]) in quite a direct and easy way; in particular, we avoid here the Newlander-Nirenberg structural theorem used in [**16**]. Our filtration is, in some way, easier to get than the classical one: indeed, due to the forementionned canonical filtration of q-difference modules, our systems admit a natural triangularisation which is independent of the choice of a Stokes direction and of the domination order of exponentials (here replaced by theta functions). Also, our filtration has a much nicer structure than the classical Gevrey filtration since the so-called elementary sheaves of the classical theory are here replaced by holomorphic vector bundles endowed with a very simple structure over an elliptic curve (they are tensor products of flat bundles by line bundles).

On the side of what this paper does not contain, there is neither a study of confluency when q goes to 1, nor any application to Galois theory. As for the former, we hope to extend the results in [**20**] to the irregular case, but this seems a difficult matter. Only partial results by C. Zhang are presently available, on significant examples. As for the latter, it is easier to obtain as a consequence of the present results that, under natural restrictions, "canonical Stokes operators are Galoisian" like in [**9**]. However, to give this statement its full meaning, we have to generalize the results of [**21**] and to associate vector bundles to arbitrary equations. This is a quite different mood that we will develop in a forthcoming paper ([**23**]; meanwhile, a survey is given in [**24**]). Here, we give some hints in remarks 3.11 and 4.5.

Also, let us point out that there has been little effort made towards systematisation and generalisation. The intent is to get as efficiently as possible to the striking specific features of q-difference theory. For instance, most of the results about morphisms between q-difference modules can be obtained by seing these morphisms as meromorphic solutions of other modules (internal Hom) and they can therefore be seen as resulting from more general statements. These facts, evenso quite often sorites, deserve to be written. In the same way, the many regularity properties of the homological equation $X(qz)A(z) - B(z)X(z) = Y(z)$ should retain some particular attention and be clarified in the language of functional analysis. They are implicitly or explicitly present in many places in the work of C. Zhang. Last, the q-Gevrey filtration should be translated in terms of factorisation of Stokes operators, like in [**9**].

Let us now describe the organization of the paper.

Notations and conventions are given in subsection 1.2.

Section 2 deals with the recent developments of the theory of q-difference equations and some improvements. In subsection 2.1, we recall the local classification of fuchsian systems by means of flat vector bundles as it can be found in [**21**] and its easy extension to the so-called "tamely irregular" q-difference modules. We then describe the filtration by the slopes ([**22**]). In subsections 2.2, we summarize results from [**16**] about the local analytic classification of irregular q-difference systems, based on the Stokes sheaf. The lemma 2.7 provides a needed improvement about Gevrey decay; proposition 2.8 and corollary 2.10 an improvement about polynomial normal forms.

In chapter 3, we first build our main tool, the algebraic summation process (theorem 3.7). Its application to the local classification is then developed in subsection 3.2. We state there and partially prove the second main result of this paper (theorem 3.18): a q-analog of the Malgrange-Sibuya theorem for the local analytic classification of linear differential equations.

Section 4 is devoted to studying the q-Gevrey filtration of the Stokes sheaf and proving the theorem 3.18. In subsection 4.1, we show how conditions of flatness (otherwise said, of q-Gevrey decay) of solutions near 0 translate algebraically and how to provide the devissage for the Stokes sheaf of a "tamely irregular" module. In subsection 4.2 we derive some cohomological consequences and we finish the proof of the theorem 3.18. Finally, in subsection 4.3, we sketch the Stokes sheaf of a general module.

The symbol □ indicates the end of a proof or the absence of proof if considered straightforward. Theorems, propositions and lemmas considered as "prerequisites" and coming from the quoted references are not followed by the symbol □.

Acknowledgements. — The present work is directly related with the paper [**16**], written in collaboration with Jean-Pierre Ramis and with Changgui Zhang. It has been a great pleasure to talk with them, confronting very different points of view and sharing a common excitement.

The epigraph at the beginning of this paper is intended to convey the happiness of wandering and daydreaming in Jean-Pierre Ramis' garden; and the overwhelming surprise of all its bifurcations. Like in Borges' story, pathes fork and then unite, the same landscapes are viewed from many points with renewed pleasure. This strong feeling of the unity of mathematics without any uniformity is typical of Jean-Pierre.

1.2. Notations and general conventions. — We fix once for all a complex number $q \in \mathbf{C}$ such that $|q| > 1$. We then define the automorphism σ_q on various rings, fields or spaces of functions by putting $\sigma_q f(z) = f(qz)$. This holds in particular for the field $\mathbf{C}(z)$ of complex rational functions, the ring $\mathbf{C}\{z\}$ of convergent power series and its field of fractions $\mathbf{C}(\{z\})$, the ring $\mathbf{C}[[z]]$ of formal power series and its field of fractions $\mathbf{C}((z))$, the ring $\mathcal{O}(\mathbf{C}^*, 0)$ of holomorphic germs and the field $\mathcal{M}(\mathbf{C}^*, 0)$ of meromorphic germs in the punctured neighborhood of 0, the ring $\mathcal{O}(\mathbf{C}^*)$ of holomorphic functions and the field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions on $\mathbf{C}^*$; this also holds for all modules or spaces of vectors or matrices over these rings and fields.

For any such ring (resp. field) R, the σ_q-invariants elements make up the subring (resp. subfield) R^{σ_q} of constants. For instance, the field of constants of $\mathcal{M}(\mathbf{C}^*, 0)$ or that of $\mathcal{M}(\mathbf{C}^*)$ can be identified with a field of elliptic functions, the field $\mathcal{M}(\mathbf{E}_q)$ of meromorphic functions over the complex torus (or elliptic curve) $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. We shall use heavily the theta function of Jacobi defined by the following equality:

$$\theta_q(z) = \sum_{n \in \mathbf{Z}} q^{-n(n+1)/2} z^n.$$

This function is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete q-spiral $[-1; q]$, where we write $[a; q] = aq^{\mathbf{Z}}$, $(a \in \mathbf{C}^*)$. It satisfies the functional equation: $\sigma_q \theta_q = z\theta_q$. We shall also use its multiplicative translates $\theta_{q,c}(z) = \theta_q(z/c)$ (for $c \in \mathbf{C}^*$); the function $\theta_{q,c}$ is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete q-spiral $[-c; q]$ and satisfies the functional equation: $\sigma_q \theta_{q,c} = \frac{z}{c}\theta_{q,c}$.