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MONODROMY PRESERVING DEFORMATION AND DIFFERENTIAL GALOIS GROUP I

by

Hiroshi Umemura

For J.-P. Ramis on the occasion of his 60th birthday

Abstract. — In 1914, J. Drach interpreted in terms of his infinite dimensional differential Galois theory R. Fuchs' work on the monodromy preserving deformation and the sixth Painlevé equation. This note of Drach contains a quite original idea but it is difficult to understand. We analyze his note by our infinite dimensional differential Galois theory. We get non-trivial examples of which we can calculate our Galois group.

***Résumé* (Déformation isomonodromique et groupe de Galois différentiel).** — En 1914, J. Drach interpréta le travail de R. Fuchs sur les déformations isomonodromiques et la sixième équation de Painlevé en termes de sa théorie de Galois de dimension infinie. La note de Drach contient une idée très originale mais difficile à comprendre. Nous analysons sa note en appliquant notre théorie de Galois différentielle de dimension infinie. Cela nous donne des exemples non triviaux dont nous pouvons calculer notre groupe de Galois.

1. Introduction

Today, there are a variety of ways of defining the Painlevé equations. Most of them are unimaginable from the original definition.

(1) Historically the origin of the Painlevé equations goes back to the pursuit of special functions defined by algebraic differential equations of the second order. Around 1900 Painlevé succeeded in classifying algebraic differential equations $y'' = F(t, y, y')$ without movable singular points, where F is a rational function of t, y and y' and t is the independent variable so that $y' = dy/dt$ and $y'' = d^2y/dt^2$. The property of being free from the movable singularities is nowadays called the Painlevé property. After he classified the equations satisfying the condition, Painlevé then threw away those

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equations that he could integrate by the so far known functions and thus he arrived at the list of the six Painlevé equations. This is the first definition of the Painlevé equations. It is, however, very lucky that he could discover the Painlevé equations in this manner.

(2) In 1907, R. Fuchs discovered that the sixth Painlevé equation describes a monodromy preserving deformation of a second order ordinary linear equation $y'' = p(x)y$. Later R. Garnier generalized this for the other Painlevé equations.

(3) In our former work [**5**], we showed that we can recover the second Painlevé equation form a rational surface with a rational double point. We can regard this as an algebro-geometric definition of the second Painlevé equation.

(4) Masatoshi Noumi and Yasuhiko Yamada interpreted theory of Painlevé equations form the view point of Kač-Moody Lie algebra. They not only uniformly reviewed the theory of τ function of the Painlevé equations but also generalized the Painlevé equations in a natural frame work.

(5) There is another definition due to J. Drach [**1**] in 1914. He asserts the equivalence of the following two conditions for a function $\lambda(t)$.

(i) $\lambda(t)$ satisfies the sixth Painlevé equation.

(ii) The dimension of the Galois group of a non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}$$

is finite.

In the second condition, the Galois group of general algebraic differential equation is involved. Namely the second condition depends on his infinite dimensional differential Galois theory, which has been an object of discussion since he proposed it in his thesis in 1898.

In this note, we apply our infinite dimensional Galois theory of differential equations [**7**] to study the result of J. Drach. We prove that the first condition (i) implies the second (ii).

Theorem 1.1. — *Let $\lambda(t)$ be a function of t satisfying the sixth Painlevé equation. Let $K = \mathbf{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies*

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}.$$

Then the Galois group $\mathrm{InfGal}(L/K)$ *is at most of dimension* 3.

Remark 1.2. — We can expect that generically the dimension of $\mathrm{InfGal}(L/K)$ is 3. Yet inequality $\dim \mathrm{InfGal}(L/K) < 3$ may occur. So it is important to determine the solutions λ of the sixth Painlevé equation and the corresponding $\mathrm{InfGal}(L/K)$ such that $\dim \mathrm{InfGal}(L/K) < 3$.

For the first Painlevé equation, we can prove a more precise result. However, this still relies on a statement about constant fields, called Proposition 5.3 below, which will be proven in [**8**].

***Theorem 1.3* (assuming Proposition 5.3 in § 5).** — *Let $\lambda(t)$ be a function of t satisfying the first Painlevé equation $\lambda'' = 6\lambda^2 + t$. Let $K = \mathbf{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies q*

$$\frac{dy}{dt} = \frac{1}{2}\frac{1}{y - \lambda(t)}$$

Then the Galois group

$$\mathrm{InfGal}(L/K) \simeq \widehat{\mathrm{SL}}_{2L^\natural}.$$

Remarks 1.4. As the proof of the Theorems shows, it is difficult to imagine how to deduce the condition (i) from (ii).

The assertion of Drach should be properly understood otherwise we would have counter examples. In fact, the second condition (ii) is closed under the specialization of the function $\lambda(t)$, whereas the first (i) is not so. Hence the first condition (i) should be replaced by

(i)' The function $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI} or a degeneration of P_{VI}.

Why are the Theorems interesting? Because the Galois group, which is a formal group of infinite dimension in general, is very difficult to calculate. We have only two types of examples where we can calculate the Galois group. (1) If L/K is a strongly normal extension in the sense of Kolchin which is his generalization of classical Galois extension so that the Galois group $G := \mathrm{Gal}(L/K)$ of the extension is an algebraic group, then $\mathrm{InfGal}(L/K) = \widehat{G}$ and (2) for differential field extension $L = K(y)/K$ such that y is a solution of a Riccati equation with coefficients in K, $\mathrm{InfGal}(L/K)$ is a formal subgroup of $\widehat{\mathrm{SL}}_2$ (*cf.* Theorem (5.16), [**7**]).

Since we can prove only one direction of the assertion of Drach, our result is not satisfactory in the sense that it does not give us a new definition of the Painlevé equation. It offers us, however, highly non-trivial examples of differential field extensions of which we can calculate our Galois group.

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2. Review of R. Fuchs' paper

R. Fuchs studied a monodromy preserving deformation of a linear differential equation $d^2y/dx^2 = p(x)y$. Namely he considered a system of linear equations

$$(1)\qquad \begin{cases} \dfrac{\partial^2 y_i}{\partial x^2} = p y_i, \\ \dfrac{\partial y_i}{\partial t} = B y_i - A\dfrac{\partial y_i}{\partial x}, \end{cases} \qquad \text{for } i = 1,\, 2,$$

where

$$p = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{e}{(x-\lambda)^2} + \cdots$$

and we assume that λ is not a function of x but it is a function of t, *i.e.*, $\partial\lambda/\partial x = 0$. y_1 and y_2 are linearly independent solutions. The integrability of the system (1) implies

$$A(x,\, t) = \frac{x(x-1)(t-\lambda)}{t(t-1)(x-\lambda)} \quad \text{and} \quad B(x,\, t) = \frac{1}{2}\frac{\partial A}{\partial x}$$

and $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI}.

Where does the non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(y-\lambda)}{t(t-1)(t-\lambda)}$$

in Theorem 1.1 come from?

Lemma 2.1. — *We may assume that the Wronskian*

$$W_r = \begin{vmatrix} y_1 & y_2 \\ \partial y_1/\partial x & \partial y_2/\partial x \end{vmatrix} = 1.$$

Proof. — It is an exercise to check $\partial W_r/\partial t = \partial W_r/\partial x = 0$ so that W_r is a constant. It is sufficient to replace y_i by $(1/\sqrt{W_r})y_i$ for $i = 1, 2$. □

From now on we write T for t, W for x so that we consider the system

$$(2)\qquad \begin{cases} \dfrac{\partial^2 y_i}{\partial W^2} = p y_i, \\ \dfrac{\partial y_i}{\partial T} = B(W,\, T) y_i - A(W, T)\dfrac{\partial y_i}{\partial W}, \end{cases} \qquad \text{for } i = 1,\, 2,$$

Lemma 2.2. — *If we set $y = y_2/y_1$, then we have*

$$\begin{cases} \dfrac{\partial y}{\partial W} = \dfrac{1}{y_1^2}, \\ \dfrac{\partial y}{\partial T} = -A\dfrac{1}{y_1^2}. \end{cases}$$

Proof. — This is a consequence of Lemma 2.1. □