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Astérisque, tome 296 (2004), p. 43-53
<http://www.numdam.org/item?id=AST_2004__296__43_0>

Astérisque
296, 2004, p. 43–53

GALOIS REPRESENTATIONS, DIFFERENTIAL EQUATIONS, AND q-DIFFERENCE EQUATIONS : SKETCH OF A p-ADIC UNIFICATION

by

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Abstract. — This is a broad introduction to the following, more technical, paper [**AdV**]. We explain how [**AdV**] relates to two major themes of J.-P. Ramis' work, which eventually become unified in the p-adic world.

***Résumé* (Représentations galoisiennes, équations différentielles et aux q-différences: esquisse d'une unification p-adique)**

Ce texte est une introduction développée à l'article suivant, plus technique [**AdV**]. Nous expliquons comment [**AdV**] est lié à deux thèmes majeurs de l'œuvre de J.-P. Ramis, et comment ceux-ci trouvent leur unification en passant au monde p-adique.

Introduction

Two remarkable analogies haved played an important role in Jean-Pierre Ramis' work:

– the analogy between linear complex differential equations and coverings in characteristic p (reported in D. Bertrand's contribution to this volume),

– the analogy between linear differential equations and q-difference equations (reported in J. Sauloy's contribution).

Our aim is to explain the analogs of these analogies in the p-adic world. We will see that once transposed into that context, these analogies become much more precise, and eventually lead to some equivalences of categories!

2000 ***Mathematics Subject Classification.*** — Primary 12H25; Secondary 34A30, 11S80, 14H30, 39A13, 11S15.

Key words and phrases. — Differential equations, q-difference equations, coverings, wild singularities, local Galois representation, overconvergence, p-adic local monodromy.

1. A mysterious analogy: linear complex differential equations and coverings in characteristic p, tame and wild

1.1. A dictionary. — This analogy grew out of discussions between J.-P. Ramis and M. Raynaud during the "Nuit de la Musique 1993"[1]. Let us recall it in the form of a "dictionary":

Differential side	Characteristic-p side
$X = \overline{X} \smallsetminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \smallsetminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module / X	unramified Galois covering of X
singular point (in S)	branch point (in S)
regular singular point	tame branch point, *i.e.*, the ramification index at s is prime to p
irregular singular point	wild branch point
local differential Galois group at $s \in S$	inertia group at $s \in S$
(global) differential Galois group G (a linear alg. group / $\mathbb{C}$)	covering group G (a finite group)
torus in G	*p-Sylow subgroup of* G
$L(G)$: normal subgroup generated by all tori	$p(G)$: normal subgroup generated by all p-Sylow's
monodromy map $\mu : \pi_1(X) \to G/L(G)$	monodromy map $\mu : \pi_1^{(p')}(X) \to G/p(G)$
μ has Zariski-dense image (Ramis condition for the existence of a diff. module on X with diff. Galois group G, all singularities $s \in S$ being regular but one)	μ is surjective (Harbater condition for the existence of an unramified G-covering of X, all branch points $s \in S$ being tame but one).

Comment. — In the right-hand column, $\overline{\mathbf{F}}_p$ denotes a fixed algebraic closure of the field $\mathbf{F}_p$ with p-elements, and $\pi_1^{(p')}(X)$ denotes the profinite group which classifies unramified coverings of X of degree prime to p, *i.e.*, the prime-to-p quotient of Grothendieck's algebraic fundamental group $\pi_1(X)$ of X. According to Grothendieck,

[1] Older sources, in the ℓ-adic context, will be evoked in the next section.

$\pi_1^{(p')}(X)$ is a free prime-to-p profinite group on $2g+|S|-1$ generators (g denotes the genus of $\overline{X}$ and S is assumed to be non-empty).[2]

1.2. ℓ-adic linearized variant ($\ell \neq p$). — There is a somewhat older and more standard version of this dictionary (*cf. e.g.* the end of [**K**]) in which objects in the right-hand column are replaced by more linear ones (in fact $\mathbb{Z}_\ell$-linear[3] ones, for some fixed (but arbitrary) prime number $\ell \neq p$). It consists essentially in considering at once the whole tower of unramified coverings of X of degree a power of ℓ. In that way, finite groups are replaced, in the right-hand column, by ℓ-adic Lie groups, or even by algebraic groups over $\mathbb{Q}_\ell$ (by taking a suitable algebraic envelope).

Differential side	Characteristic-p side
$X = \overline{X} \setminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \setminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module M on X	lisse ℓ-adic sheaf $\mathcal{L}$ on X (ℓ-adic continuous representation of $\pi_1(X)$)
differential Galois group (an algebraic group / $\mathbb{C}$)	monodromy group (image of $\pi_1(X)$ or its Zariski closure, an algebraic group / $\mathbb{Q}_\ell$)
local differential Galois group	image of inertia group $\mathcal{I}$ (or its Zariski closure)
de Rham cohomology groups $H^i_{\mathrm{dR}}(X, M)$	étale cohomology groups $H^i_{\text{ét}}(X, \mathcal{L})$
$\chi(M) = \sum(-1)^i \dim H^i_{\mathrm{dR}}(X, M)$	$\chi(\mathcal{L}) = \sum(-1)^i \dim H^i_{\text{ét}}(X, \mathcal{L})$
Deligne-Malgrange irregularity $\mathrm{irr}(M, s)$ *at* s	*Swan conductor* $\mathrm{sw}(M, s)$ *at* $s \in S$
Deligne's formula for $\chi(M)$ in terms of $\mathrm{rk}\, M$ and irregularities	Grothendieck's formula for $\chi(\mathcal{L})$ in terms of $\mathrm{rk}\, M$ and Swan conductors.

[2] Referee's remark. Earlier presentations of the Ramis-Raynaud dictionary can be found in M. van der Put's Bourbaki talk: Recent work on differential Galois theory (Exposé 849, Astérisque 252 (1998), 341-367), as well as in van der Put and Singer's book *Galois Theory of Linear Differential Equations*, Springer-Verlag (2003).

[3] Recall that the ring of ℓ-adic integers $\mathbb{Z}_\ell$ is the limit of the system $\cdots \to \mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to \cdots \mathbb{Z}/\ell\mathbb{Z} = \mathbf{F}_\ell$, so that any ℓ-adic integer can be expressed as a series $\sum_0^\infty a_n \ell^n$ where $a_n \in \{0, 1, \ldots, \ell-1\}$. The field of fractions of $\mathbb{Z}_\ell$ is $\mathbb{Q}_\ell = \mathbb{Z}_\ell[\frac{1}{\ell}]$. In the sequel, we denote by $\overline{\mathbb{Q}}_\ell$ a fixed algebraic closure of $\mathbb{Q}_\ell$.

1.3. The ℓ-adic local monodromy theorem ($\ell \neq p$). — Let us recall the structure of the absolute Galois groups which play a role in the "characteristic-p side". We now assume that $k = \mathbf{F}_{p^n} \subset \overline{k} = \overline{\mathbf{F}}_p$ is the field with p^n elements. Then,

$$G_k := \mathrm{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}} = \prod_{\ell'} \mathbb{Z}_{\ell'} \quad \text{and}$$

$G_{k((x))} := \mathrm{Gal}\left(k((x))^{\mathrm{sep}}/k((x))\right)$ can be unscrewed via two exact sequences:

$$1 \longrightarrow \mathcal{I} \longrightarrow G_{k((x))} \longrightarrow G_k \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I} \longrightarrow \mathbb{Z}_\ell \times \prod_{\ell' \neq p, \ell} \mathbb{Z}_{\ell'} \longrightarrow 1$$

where $\mathcal{I} = G_{\overline{k}((x))}$ is the *inertia group*, and $\mathcal{P}$ is a pro-p-group called the *wild inertia group*.

This reflects the fact that in contrast to the char. 0 case, the algebraic closure of $\overline{k}((x))$ contains many more elements than just Puiseux series. For instance, roots z of the Artin-Schreier equation $z^{-p} - z^{-1} = x^{-1}$ cannot be expressed as Puiseux series.

Correspondingly, one has a tower of Galois extensions

$$k((x)) \subset \overline{k}((x)) \overset{\text{tame}}{\subset} \bigcup_{p \nmid n} \overline{k}((x^{1/n})) \overset{\text{wild}}{\subset} (k((x)))^{\mathrm{sep}},$$

with respective Galois groups $G_k, \mathcal{I}/\mathcal{P}$ and $\mathcal{P}$.

***Theorem 1.1* (Grothendieck [G]).** — *Every ℓ-adic representation of $G_{k((x))}$ is* quasi-unipotent, *i.e., a suitable open subgroup of $\mathcal{I}$ acts (through its quotient in $\mathbb{Z}_\ell$) by unipotent matrices.*

This can also be formulated, in the "Tannakian vein", as an equivalence of $\otimes$-categories

$$Rep_{\overline{\mathbb{Q}}_\ell}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{continuous } \overline{\mathbb{Q}}_\ell\text{-reps. of } \mathcal{I} \text{ which extend to reps. of } G_F\}$$

where $\mathcal{I}$ appears in the left-hand side as a constant group-scheme (and representations are understood in the scheme-theoretic sense), and in the right-hand side as a profinite topological group.

2. The p-adic analog of this analogy. An equivalence of categories.

2.1. Some motivation. Frobenius and overconvergence.— At least two aspects of the above dictionary 1.2 are unsatisfactory: the arbitraryness of the auxiliary prime number ℓ, and the very different natures of the cohomologies occurring in the left-hand (De Rham) and right-hand (étale) columns.

Both drawbacks would disappear, and the analogy would become much closer, if one could replace ℓ by p, and étale cohomology by some appropriate cohomology of De Rham type.