Astérisque

Yves André
Lucia Di Vizio

q-difference equations and p-adic local monodromy

Astérisque, tome 296 (2004), p. 55-111
<http://www.numdam.org/item?id=AST_2004__296__55_0>

Astérisque
296, 2004, p. 55–111

q-DIFFERENCE EQUATIONS AND p-ADIC LOCAL MONODROMY

by

Yves André & Lucia Di Vizio

Abstract. — We present a p-adic theory of q-difference equations over arbitrarily thin annuli of outer radius 1. After a detailed study of rank one equations, we consider higher rank equations and prove a local monodromy theorem (a q-analog of Crew's quasi-unipotence conjecture). This allows us to define, in this context, a canonical functor of "confluence" from q-difference equations to differential equations, which turns out to be an equivalence of categories (in the presence of Frobenius structures).

Résumé **(Équations aux q-différences et monodromie p-adique).** — Nous présentons une théorie p-adique des équations aux q-différences sur des couronnes arbitrairement minces de rayon extérieur 1. Après une étude détaillée des équations de rang 1, nous nous penchons sur le cas de rang supérieur et nous démontrons un théorème de monodromie locale (un q-analogue de la conjecture de quasi-unipotence de Crew). Cela nous permet de définir, dans ce contexte, un foncteur canonique de « confluence » des équations aux q-différences vers les équations différentielles, qui s'avère être une équivalence de catégories (en présence de structures de Frobenius).

Introduction

In the context of p-adic differential equations, the expression "local theory" occurs in two different senses. In the naive sense, it refers to the study of the behaviour of solutions in a small punctured disk around a given singularity. This theory has been reasonably well-understood for a long time[(1)].

On the other hand, according to some insights of Dwork and Grothendieck, the geometrically relevant p-adic differential equations are those which admit analytic solutions in all non-singular open unit disks, and which extend a little inside the singular disks. They should be understood as objects (cohomological coefficients) belonging to geometry in characteristic p. It is then consistent with this viewpoint

2000 ***Mathematics Subject Classification.*** — Primary 39A13; Secondary 33D05, 12H50.
Key words and phrases. — p-adic monodromy, q-difference equations, p-adic representation.

(1) Although by no means completely understood, *cf.* for instance the problems raised by Ramero's theory [**Ra98**] in its differential variant.

to call "local theory" the study of the behaviour of solutions in arbitrarily thin annuli with outer radius 1 contained in singular open unit disks[(2)].

In this sense, the local theory of p-adic differential equations has been developped first by Robba (in rank one), then by Christol and Mebkhout (in arbitrary rank), and has recently reached full maturity with the proof of the so-called local monodromy theorem (Crew's quasi-unipotence conjecture) which provides a bridge toward the theory of p-adic Galois representations.

The objective of this paper is to set up a local theory of p-adic q-difference equations, parallel to the differential theory, and to put a link forward between the two theories.

* * *

In the history of the theory of p-adic differential equations, going from the rank 1 case to arbitrary rank has been a difficult step. This is due in part to the fact that the study of rank 1 p-adic differential equations indulges fairly down-to-earth methods (*cf.* for instance [**R85**], [**CC96**]). In the first part of the paper we develop an analogous theory for p-adic q-difference equations of rank 1. The techniques employed are inspired by the differential case and, due to their explicit and direct nature, bring to the fore the relationship with differential equations. In fact, we construct a *canonical deformation functor* from the category of p-adic differential equations of rank 1 to the category of p-adic q-difference equations, which we describe explicitly.

The first part is organized as follows. In § 1 we recall some basic facts of p-adic q-difference algebra proved in [**DV03**]. In § 2 we prove some properties of the q-exponential function which play a significant role in the sequel. Sections § 3 and § 4 contain a q-analog of Dwork-Robba's criterion of solvability and its application to q-difference equations of rank 1 with meromorphic coefficient. The results in § 4 are used in the next section to show that one can actually reduce the study of rank-one q-difference equations analytic over an arbitrary thin annulus of outer radius 1, to the study of rank 1 q-difference equations with polynomial coefficient. This reduction is crucial for the characterization of q-difference equations with Frobenius structure (*cf.* § 6). We finish the first part by proving that for a q-difference equation having a Frobenius structure is equivalent to being a "deformation" of a differential equation with strong Frobenius structure (*cf.* § 7). From there, we obtain the p-adic monodromy theorem in the rank 1 case and the deformation functor (*cf.* § 8).

There are two appendices, the first one being devoted to the Frobenius structure of the q-exponential series. In the second one, we give a q-analog of Dwork's approach to the p-adic gamma function via the Frobenius structure of so-called exponential modules.

* * *

[(2)] See the previous paper [**A**] for more detail and perspective, and for the apparatus of analogies which motivates the present paper.

In the second part, we consider q-difference modules M of arbitrary rank over the "Robba ring" $\mathcal{R}$ of analytic functions on an arbitrarily thin annulus of outer radius 1. We prove the local monodromy theorem for those q-difference modules which admit a Frobenius structure: there exists a finite étale extension $\mathcal{R}'/\mathcal{R}$ coming from characteristic p, such that $M \otimes_{\mathcal{R}} \mathcal{R}'[\log x]$ becomes a trivial q-difference module (*cf.* § 14.2, § 14.3 for various equivalent precise statements). We follow K. Kedlaya's approach to the p-adic local monodromy theorem in the differential case, proving along the way a q-analog of Tsuzuki's theorem on unit-root objects.

This second part is organized as follows. We first discuss finite étale extensions $\mathcal{R}'/\mathcal{R}$ coming from characteristic p, and how the q-difference operator d_q extends to $\mathcal{R}'$ (the lack of an explicit expression for this extended operator leads to many technical difficulties in the sequel). We then introduce and investigate two notions of Frobenius structures for q-difference modules: the strong Frobenius structure (analogous to its differential counterpart), and the confluent weak Frobenius structure (which yields a sequence of q^{p^n}-difference modules converging to a differential module with Frobenius structure).

In § 13, we analyse q-difference modules over $\mathcal{R}$ with overconvergent (strong) Frobenius structure of slope 0. As in Tsuzuki's theorem, they arise from finite p-adic representations of the inertia group of a local field of characteristic p.

We then prove three versions of the theorem of quasi-unipotence for q-difference modules over $\mathcal{R}$ which admit a strong Frobenius structure. We also show that such q-difference modules have a confluent weak Frobenius structure.

This gives rise to a canonical functor of "confluence" between such q-difference modules (M, Σ_q), and differential modules over $\mathcal{R}$ which admit a strong Frobenius structure, which has a canonical quasi-inverse (15.1, 15.2). More precisely, for any such (M, Σ_q), there is a canonical sequence of $q^{p^{is}}$-difference structures on the $\mathcal{R}$-module M (for fixed s and with $i \to \infty$, so that $q^{p^{is}} \to 1$), related to each other by Frobenius, and which converges to a differential structure on M.

PART I
RANK 1

1. Generalities on p-adic q-difference equations of rank 1

1.1. The q-difference algebra of analytic functions over an annulus

Let K be a field of characteristic zero, complete with respect to a non archimedean absolute value $| \ |$, with residue field k of characteristic $p > 0$. We denote by $\mathcal{O}_K$ its ring of integers and we assume that the absolute value is normalized by $|p| = p^{-1}$.

For any interval $I \subset \mathbb{R}_{\geqslant 0}$ we consider the K-algebra $\mathcal{A}_K(I)$ of analytic functions with coefficients in K on the annulus $\mathcal{C}_K(I) = \{x \in K : |x| \in I\}$:

$$\mathcal{A}_K(I) = \left\{ \sum\nolimits_{n\in\mathbb{Z}} a_n x^n : a_n \in K;\ \lim_{n\to\pm\infty} |a_n|\rho^n = 0\ \forall \rho \in I \right\}.$$

We denote by $\mathcal{M}_K(I)$ its field of fractions (the field of meromorphic functions on $\mathcal{C}_K(I)$), and by $\mathcal{B}_K(I)$ the subring of bounded elements of $\mathcal{A}_K(I)$. The theory of Newton polygons shows that every invertible analytic function on $\mathcal{C}(I)$ is bounded, so that $\mathcal{A}_K(I)^* = \mathcal{B}_K(I)^*$. We will omit the subscript K when there is no ambiguity.

We fix once and for all an element $q \in K$, such that $|1-q| < 1$ and that q is not a root of unity. The algebra $\mathcal{A}(I)$ has a natural structure of a q-difference algebra. This means that the homeomorphism

$$\begin{aligned} \mathcal{C}(I) &\longrightarrow \mathcal{C}(I) \\ x &\longmapsto qx \end{aligned}$$

induces a K-algebra isomorphism

$$\begin{aligned} \sigma_q : \mathcal{A}(I) &\longrightarrow \mathcal{A}(I) \\ f(x) &\longmapsto f(qx) \end{aligned}$$

Similarly for $\mathcal{M}(I)$ and $\mathcal{B}(I)$.

1.2. The q-derivation. — To the operator σ_q one associates a "twisted derivation" d_q defined by

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

which satisfies the twisted Leibniz Formula:

$$d_q(fg)(x) = f(qx) d_q(g)(x) + d_q(f)(x) g(x). \tag{1}$$

For any pair of integers $n \geqslant i \geqslant 1$ and any $f, g \in \mathcal{M}(I)$ the q-derivation d_q verifies:

(2) $d_q x^n = [n]_q x^{n-1}$, where $[n]_q = 1 + q + \cdots + q^{n-1} = \dfrac{q^n - 1}{q-1}$;

(3) $\dfrac{d_q^n}{[n]_q^!} x^i = \dbinom{n}{i}_q x^{n-i}$, where $[0]_q^! = 1$, $[n]_q^! = [n]_q [n-1]_q^!$ and $\dbinom{n}{i}_q = \dfrac{[n]_q^!}{[i]_q^! [n-i]_q^!}$;

(4) $d_q^n(fg)(x) = \displaystyle\sum_{j=0}^{n} \binom{n}{j}_q d_q^{n-j}(f)(q^j x) d_q^j(g)(x)$.

1.3. q-difference equations. — Let us now consider a *q-difference equation of rank* 1 *with coefficients in* $\mathcal{M}(I)$:

$$y(qx) = a(x) y(x), \quad a(x) \in \mathcal{M}(I). \tag{5}$$