Astérisque

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Astérisque, tome 297 (2004), p. 117-166
<http://www.numdam.org/item?id=AST_2004__297__117_0>

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297, 2004, p. 117–166

ON THE STOKES GEOMETRY OF HIGHER ORDER PAINLEVÉ EQUATIONS

by

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Abstract. — We show several basic properties concerning the relation between the Stokes geometry (*i.e.*, configuration of Stokes curves and turning points) of a higher order Painlevé equation with a large parameter and the Stokes geometry of (one of) the underlying Lax pair. The higher-order Painlevé equation with a large parameter to be considered in this paper is one of the members of P_J-hierarchy with J = I, II-1 or II-2, which are concretely given in Section 1. Since we deal with higher order equations, the Stokes curves may cross; some anomaly called the Nishikawa phenomenon may occur at the crossing point, and in this paper we analyze the mechanism why and how the Nishikawa phenomenon occurs. Several examples of Stokes geometry are given in Section 5 to visualize the core part of our results.

Résumé **(Sur la géométrie de Stokes des équations de Painlevé d'ordre supérieur)**

Nous exhibons plusieurs propriétés fondamentales liant, d'une part, la géométrie de Stokes (*i.e.*, la configuration des courbes de Stokes et des points tournants) d'une équation de Painlevé d'ordre supérieur à grand paramètre et, d'autre part, la géométrie de Stokes de l'une des paires de Lax sous-jacentes. L'équation de Painlevé d'ordre supérieur à grand paramètre considérée est l'une des équations de la hiérarchie P_J pour J = I, II-1 ou II-2 que nous détaillons dans le paragraphe 1. Les équations étant d'ordre supérieur leurs lignes de Stokes peuvent se croiser et l'anomalie connue sous le nom de « phénomène de Nishikawa » peut se produire aux points de croisement. Nous analysons le mécanisme par lequel ce phénomène de Nishikawa apparaît. Plusieurs exemples de géométrie de Stokes sont donnés dans le paragraphe 5 en vue d'une visualisation de la partie centrale de nos résultats.

2000 ***Mathematics Subject Classification.*** — Primary 34M60; Secondary 34E20, 34M40, 33E17.
Key words and phrases. — Exact WKB analysis, higher order Painlevé equations.

This research is supported by JSPS Japan-Australia Research Cooperative Program and partially by JSPS Grant-in-Aid No. 14340042. T. Koike and Y. Takei are respectively supported also by JSPS Grant-in-Aid No. 15740088 and No. 13640167.

0. Introduction

This paper is the first of a series of our papers on the exact WKB analysis of higher order Painlevé equations. For the sake of the clarity and the uniformity of the description we restrict our consideration in this paper to the $P_{\mathrm{I}}, P_{\mathrm{II}\text{-}1}$ and $P_{\mathrm{II}\text{-}2}$ hierarchies with a large parameter η, which are described explicitly in Section 1. Although these hierarchies are basically the same as those discussed by Shimomura ([**S2**]), Gordoa-Pickering ([**GP**]) and Gordoa-Joshi-Pickering ([**GJP**]), we need to appropriately introduce a large parameter η in their coefficients together with the underlying systems of linear differential equations (the so-called Lax pairs) so that we may develop the WKB analysis of the hierarchies in question. As is evident in the series of papers ([**KT1, AKT2, KT2, T1**]; see [**KT3**] for their résumé), the relations between the Stokes geometry for (one of) the Lax pair and the appropriately defined Stokes geometry for the Painlevé equation play the key role in the WKB analysis of the traditional Painlevé equations, *i.e.*, the second order differential equations first studied by Painlevé and Gambier. One of our main purposes of this paper is to show that the relations observed for the traditional Painlevé equations remain to hold for each member in the Painlevé hierarchies considered in this paper (Section 2). Another main purpose of this paper is to analyze why the novel and interesting phenomena numerically discovered by one of us (Y.N.) should occur in our context (Section 3). To analytically detect where the phenomena (the so-called Nishikawa phenomena) are observed, we introduce the notion of new Stokes curves in Section 4. In Section 5 we present several illuminating examples of Stokes geometry for higher order Painlevé equations and the Stokes geometry of their underlying Lax pair. Appendix A gives a proof of some properties of auxiliary functions $\mathcal{K}_j$ and K_j used in Sections 1 and 2 to write down the $P_{\mathrm{II}\text{-}1}$-hierarchy with a large parameter. In Appendix B we note that the P_{I}-hierarchy with a large parameter is equivalent to a hierarchy discussed by Gordoa and Pickering ([**GP**]) if a large parameter is appropriately introduced.

As the discussion of [**KT1**] etc. uses a Lax pair of single differential equations, the results there may look pretty different from the results in this paper, where a Lax pair of 2×2 systems is used, that is, the framework of Flaschka-Newell ([**FN**]) and Jimbo-Miwa ([**JM**]) is used instead of the framework of Okamoto ([**O**]); in particular, the apparent singularities which played an important role in [**KT1**] etc. do not appear in this paper. Hence we end this introduction with briefly recalling the geometric results in [**KT1**] which are reformulated for a Lax pair of matrix equations. For the sake of simplicity we consider only the first Painlevé equation. Thus, following [**JM**], we start with the following Lax pair:

$$(0.1) \qquad \begin{cases} \Big(\dfrac{\partial}{\partial x} - \eta A\Big)\psi = 0, & (0.1.\mathrm{a}) \\ \Big(\dfrac{\partial}{\partial t} - \eta B\Big)\psi = 0, & (0.1.\mathrm{b}) \end{cases}$$

where

$$A = \begin{pmatrix} v(t,\eta) & 4(x-u(t,\eta)) \\ x^2+u(t,\eta)x+u(t,\eta)^2+t/2 & -v(t,\eta) \end{pmatrix} \tag{0.2}$$

and

$$B = \begin{pmatrix} 0 & 2 \\ x/2+u(t,\eta) & 0 \end{pmatrix}. \tag{0.3}$$

That is, we consider an isomonodromic deformation (with respect to the variable t) of the first matrix equation (0.1.a); the second equation (0.1.b) explicitly describes this deformation. To obtain (0.1) we have introduced a large parameter η to the equation (C.2) of [**JM**, p. 437] so that the resulting compatibility condition may become the first Painlevé equation with a large parameter η in [**KT1**] etc. We have also interchanged the first component and the second component of the unknown vector ψ for the sake of uniformity of presentation in this paper. The compatibility condition of the equations (0.1.a) and (0.1.b), *i.e.*,

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta(AB-BA) = 0 \tag{0.4}$$

can be readily seen to be equivalent to the following system (H_{I}):

$$(H_{\mathrm{I}}) : \begin{cases} \dfrac{du}{dt} = \eta v \\ \dfrac{dv}{dt} = \eta(6u^2+t) \end{cases}. \tag{0.5}$$

We next construct the so-called 0-parameter solution $(\widehat{u},\widehat{v})$ of (H_{I}) which has the following form:

$$\widehat{u}(t,\eta) = \widehat{u}_0(t) + \eta^{-1}\widehat{u}_1(t) + \cdots, \tag{0.6}$$

$$\widehat{v}(t,\eta) = \widehat{v}_0(t) + \eta^{-1}\widehat{v}_1(t) + \cdots. \tag{0.7}$$

It is known that, although $(\widehat{u},\widehat{v})$ is a divergent series, it is Borel summable. Note that

$$6\widehat{u}_0^2 + t = 0 \quad \text{and} \quad \widehat{v}_0 = 0 \tag{0.8}$$

hold and that $\widehat{u}_j$ and $\widehat{v}_j$ $(j \geq 1)$ are recursively determined. Substituting $(\widehat{u},\widehat{v})$ into the coefficients of A and B, we let A_0 and B_0 denote their top degree part in η, that is,

$$A_0 = \begin{pmatrix} 0 & 4(x-\widehat{u}_0(t)) \\ x^2+\widehat{u}_0(t)x+\widehat{u}_0(t)^2+t/2 & 0 \end{pmatrix}, \tag{0.9}$$

$$B_0 = \begin{pmatrix} 0 & 2 \\ x/2+\widehat{u}_0(t) & 0 \end{pmatrix}. \tag{0.10}$$

To consider the linearization of (H_{I}) at $(\widehat{u},\widehat{v})$, we set $u=\widehat{u}+\Delta u$ and $v=\widehat{v}+\Delta v$ in (0.5) and consider the part linear in $(\Delta u,\Delta v)$. (Although the terminology "linearization" used here has a completely different meaning from that used in [**JM**], we hope there is no fear of confusions; in [**JM**] etc., the linearization of (H_{I}) means the system (0.1) of linear differential equations.) Then we obtain

$$\frac{d}{dt}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix} = \eta\begin{pmatrix}0 & 1\\ 12\widehat{u} & 0\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}. \tag{0.11}$$

Let C and C_0 respectively denote

$$\begin{pmatrix}0 & 1\\ 12\widehat{u} & 0\end{pmatrix} \tag{0.12}$$

and

$$\begin{pmatrix}0 & 1\\ 12\widehat{u}_0 & 0\end{pmatrix}. \tag{0.13}$$

Concerning the matrices A_0, B_0 and C_0 we find the following several relations. First of all, (0.8) immediately entails

$$A_0 = 2(x - \widehat{u}_0)B_0. \tag{0.14}$$

This relation leads to the following

Fact A

(i) *The equation* (0.1.a) *has one double turning point* $x = \widehat{u}_0(t)$ *if* $\widehat{u}_0 \neq 0$.

(ii) *It has one simple turning point* $x = -2\widehat{u}_0(t)$ *if* $\widehat{u}_0 \neq 0$, *and this point is a turning point of the equation* (0.1.b).

Here and in what follows we use the terminology "a turning point" for a matrix equation like (0.1.a) to mean, as usual, a point where eigenvalues of its highest degree part in η (*i.e.*, the matrix A_0 in the case of (0.1.a)) merge. In other words, a turning point is a zero of the discriminant of the characteristic equation of the highest degree part, and it is said to be simple (resp. double) if it is a simple (resp. double) zero of the discriminant. We next obtain

$$12\widehat{u}_0(t)\widehat{u}_0(t)' + 1 = 0 \tag{0.15}$$

by differentiating (0.8). Then this relation proves the following

Fact B. — *The eigenvalues* $\lambda_\pm$ *of* A_0 (*i.e.*, $\pm 2(x - \widehat{u}_0)\sqrt{x + 2\widehat{u}_0}$) *and the eigenvalues* $\mu_\pm$ *of* B_0 (*i.e.*, $\pm\sqrt{x + 2\widehat{u}_0}$) *satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{0.16}$$

The following Fact C might look too trivial to note, but for the sake of later references we note it here.