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VERSAL DEFORMATION OF THE ANALYTIC SADDLE-NODE

by

Frank Loray

To Jean-Pierre Ramis for his 60th birthday

Abstract. — In the continuation of [**10**], we derive simple forms for saddle-node singular points of analytic foliations in the real or complex plane just by gluing foliated complex manifolds. We give a miniversal analytic deformation of the simplest model. We also derive a unique analytic form for those saddle-node having an analytic central manifold. By this way, we recover and generalize results earlier proved by J. Écalle by using mould theory and partially answer to some questions asked by J. Martinet and J.-P. Ramis at the end of [**11**].

Résumé **(Déformation verselle d'un nœud-col analytique).** — Dans la continuité de [**10**], nous construisons une forme normale simple pour un feuilletage analytique au voisinage d'une singularité de type nœud-col dans le plan réel ou complexe. Nous obtenons une telle forme en recollant des variétés complexes feuilletées. Nous en déduisons une déformation analytique miniverselle dans un cas simple. Nous donnons une forme unique pour un nœud-col possédant une variété centrale analytique. Nous retrouvons ainsi géométriquement et nous généralisons des résultats obtenus par J. Écalle à l'aide de la théorie des moules. Ce travail répond partiellement à des questions ouvertes posées par J. Martinet et J.-P. Ramis à la fin de [**11**].

Introduction and results

Let X be a germ of analytic vector field at the origin of $\mathbb{C}^2$

$$X = f(x,y)\partial_x + g(x,y)\partial_y, \quad f,g \in \mathbb{R}\{x,y\} \text{ or } \mathbb{C}\{x,y\}$$

having a singularity at 0: $f(0) = g(0) = 0$. Consider $\mathcal{F}$ the germ of singular holomorphic foliation induced by the complex integral curves of X near 0. A question going back to H. Poincaré is the following:

Problem. — Find local coordinates in which the foliation is defined by a vector field having coefficients as simple as possible.

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In this problem, the vector field is considered up to analytic change of coordinates and up to multiplication by a germ of analytic function. For instance, if the vector field X has a linear part (in the matrix form)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+by)\partial_x + (cx+dy)\partial_y$$

having non zero eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ with eigenratio $\lambda_2/\lambda_1 \notin \mathbb{R}$, then H. Poincaré proved that the vector field X is actually linear in convenient analytic coordinates. In this situation, the eigenvalues $\{\lambda_1, \lambda_2\}$ (resp. the eigenratio λ_2/λ_1) provide a complete set of invariants for such vector fields (resp. foliations) modulo analytic change of coordinates.

In this paper, we consider *unramified saddle-nodes*, *i.e.* foliations defined by a vector field having (exactly) one zero eigenvalue and multiplicity 2. Following H. Dulac, such a foliation is defined in convenient coordinates by a vector field of the form

$$X = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}. \tag{1}$$

and one can further *formally* reduce the vector field X to a unique form

$$X_\mu = x^2\partial_x + y\partial_y + \mu xy\partial_y, \quad \mu \in \mathbb{C} \tag{2}$$

The complete *analytic* classification of those singular points has been given by J. Martinet and J.-P. Ramis in 1982 (see [**11**] or section 1), giving rise to infinitely many invariants additional to the formal one μ above. The resulting moduli space is huge and we expect that a generic saddle-node cannot be defined by a polynomial vector field in any analytic coordinates (although this is open, as far as I know). A direct application of our recent work [**10**] provides the following

Theorem 1. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) in the form (1) above. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad f \in \mathbb{C}\{y\} \tag{3}$$

where $f'(0) = \mu$.

This statement is a particular case of a general simple analytic form independantly announced by A.D. Bruno and P.M. Elizarov for all resonant saddles ($\lambda_2/\lambda_1 \in \mathbb{Q}^-$) and saddle-nodes in 1983 (see [**3, 6**]). So far, only the case of Theorem 1 with $\mu = 0$ has been proved: it is presented by J. Écalle as an application of *resurgent functions and mould theory* at the end of [**5**], p. 535. In 1994, P.M. Elizarov made an important step toward the analytic form announced by solving in [**7**] the associate cohomological equation. One can immediately deduce from his computations that the family X_f of Theorem 1 is *miniversal* at $f \equiv 0$: the coefficients of f play the role of Martinet-Ramis' invariants at the first order. This will be rigorously stated in section 1, once we have recalled the definition (and construction) of Martinet-Ramis' invariants.

It is important to notice that the form (3) is not unique. Of course, we can modify the functional coefficient f by conjugating the vector field with an homothety $y \mapsto c{\cdot}y$, $c \in \mathbb{C}^*$. But even if we restrict to tangent-to-the-identity conjugacies, the form (3) is perhaps locally unique at X_0 ($f \equiv 0$), but not globally for the following reason. By construction (see proof of Theorem 1), the form (3) is obtained with $f(0) \neq 0$, even if the saddle node has a *central manifold* (see below). For instance, the model X_0 has also another form (3) with $f(0) \neq 0$.

From preliminary form (1), we see that $\{x = 0\}$ is an invariant curve for the vector field that we will call *strong manifold* throughout the paper. Tangent to the zero eigendirection, there is also a unique "formal invariant curve" $\{y = \varphi(x)\}$, $\varphi \in \mathbb{R}[[x]]$ or $\mathbb{C}[[x]]$, which is generically divergent. When this curve is convergent, we call it *central manifold.* A remarquable result of Martinet-Ramis' classification is that saddle-nodes having a central manifold form an analytic submanifold of codimension one (in the unramified case). For instance, saddle-nodes in the form (3) with $f(0) = 0$ have the central manifold $\{y = 0\}$. Conversely, a natural question is:

Problem. — Given a saddle-node like in Theorem 1 having a central manifold, is it possible to put it analytically into the form (3) with $f(0) = 0$ (*i.e.* simultaneously straightening the central manifold onto $\{y = 0\}$) ?

For generic μ, the answer is yes:

Theorem 2. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) like in Theorem 1 with $\mu \in \mathbb{C} - \mathbb{R}^-$. If $\mathcal{F}$ has a central manifold, then there exist local analytic coordinates in which $\mathcal{F}$ is defined by*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad \textit{with } f(0) = 0. \tag{4}$$

Moreover, this form is unique up to homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$.

In the remaining case $\mu \in \mathbb{R}^-$, we will give necessary and suffisant conditions in section 4 in terms of Martinet-Ramis' invariants (see Theorem 8), thus providing a complete answer to the question above; in the case $\mu = 0$, the condition was already given by J. Écalle in [**5**], p. 539. It turns out that these conditions are very restrictive (infinite codimension). For instance, when $\mu \in -\mathbb{N}^*$, only the saddle-nodes analytically conjugated to the formal model (2) can be normalized to the form (4). In particular, for each $\mu \in -\mathbb{N}^*$, the subfamily of those X_f satisfying $f(0) = 0$ and $f'(0) = \mu$ provides a codimension two analytically trivial deformation of the formal model (2).

Accidentally, our method to prove Theorem 2 provides in turn a simple form for saddles:

Theorem 3. — *Let $\mathcal{F}$ be a germ of saddle foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) with eigenratio $-\mu < 0$. Then there exist local analytic coordinates in which $\mathcal{F}$ is* *defined by a vector field of the form*

$$X_f = -x\partial_x + \mu(f(y)+x)y\partial_y, \quad \text{with } f(0)=1. \tag{5}$$

This latter form is not unique: for generic μ, all X_f are conjugated. For saddle-nodes having a central manifold that cannot be transformed into the form (4), it is possible to give an alternate unique form as follows.

Theorem 4. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) like in Theorem 1 having a central manifold. Let $n \in \mathbb{N}$ be such that $\mu + n \notin \mathbb{R}^-$. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xyf(x^n y)\partial_y, \quad \text{where } f(0)=\mu. \tag{6}$$

Moreover, this form is unique up to homothety $y \mapsto c\cdot y$, $c \in \mathbb{C}^*$.

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1. Martinet-Ramis' invariants

We recall the construction of [**11**]. Consider a saddle-node in Dulac preliminary form (1)

$$X = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

The Sectorial Normalization Theorem due to Hukuhara, Kimura and Matuda reads as follows. For a sufficiently small $r, \varepsilon > 0$, there exists on each of the two sectorial domains V^+ and V^-

$$V^\pm := \{|x| < r,\ |y| < r,\ 0-\varepsilon < \arg(\pm x) < \pi + \varepsilon\}$$

a unique holomorphic diffeomorphism $\Phi^\pm : V^\pm \to \Phi^\pm(V^\pm) \subset \mathbb{C}^2$ of the form $\Phi(x,y) = (x, \phi(x,y))$, which is tangent to the identity at $(0,0)$ and conjugating the saddle-node above to its formal normal form (2)

$$X_\mu := x^2\partial_x + y\partial_y + \mu xy\partial_y.$$

The model X_μ admits the first integral $H_\mu(x,y) := yx^{-\mu}e^{1/x}$. Once we have fixed determinations $H_\mu^\pm$ of H_μ on the sectors $V^\pm$ coinciding over $\{-\varepsilon < \arg(x) < +\varepsilon\}$, we immediately deduce sectorial first integrals $H^\pm := H_\mu^\pm \circ \Phi^\pm$ for the initial saddle-node.

On the overlapping $V^+ \cap V^-$, the two first integrals H^+ and H^- factorize in the following way. Over $V^0 = \{\pi-\varepsilon < \arg(x) < \pi+\varepsilon\}$, the first integrals H^+ and H^- both identify the space of leaves with a neighborhood of $0 \in \mathbb{C}$, the size of which depending on the radius r: one can write $H^- = \varphi^0 \circ H^+$ for some germ of diffeomorphism $\varphi^0 \in \mathrm{Diff}(\mathbb{C},0)$. Over the other overlapping $V^\infty = \{-\varepsilon < \arg(x) < +\varepsilon\}$, the first integrals H^+ and H^- both identify the space of leaves with $\mathbb{C}$: one can write $H^- = \varphi^\infty \circ H^+$ for some affine automorphism φ^∞ of $\mathbb{C}$. From the asymptotics of $\Phi^\pm$ and the choice