CARLOS SIMPSON

Asymptotics for general connections at infinity

Astérisque, tome 297 (2004), p. 189-231
<http://www.numdam.org/item?id=AST_2004__297__189_0>

Astérisque
297, 2004, p. 189–231

ASYMPTOTICS FOR GENERAL CONNECTIONS AT INFINITY

by

Carlos Simpson

Abstract. — For a standard path of connections going to a generic point at infinity in the moduli space M_{DR} of connections on a compact Riemann surface, we show that the Laplace transform of the family of monodromy matrices has an analytic continuation with locally finite branching. In particular, the convex subset representing the exponential growth rate of the monodromy is a polygon whose vertices are in a subset of points described explicitly in terms of the spectral curve. Unfortunately, we don't get any information about the size of the singularities of the Laplace transform, which is why we can't get asymptotic expansions for the monodromy.

***Résumé* (Asymptotique des connexions génériques à l'infini).** — Pour une courbe standard allant vers un point général à l'infini dans l'espace des modules M_{DR} des connexions sur une surface de Riemann compacte, nous montrons que le transformé de Laplace de la famille des matrices de monodromie admet un prolongement analytique avec ramification localement finie. En particulier, l'ensemble convexe qui représente la croissance exponentielle est un polygone dont les sommets sont dans un ensemble qu'on peut expliciter en termes de la courbe spectrale. Malheureusement, nous n'obtenons pas d'information sur la taille des singularités du transformé de Laplace et donc pas de développement asymptotique pour la monodromie.

1. Introduction

We study the asymptotic behavior of the monodromy of connections near a general point at ∞ in the space M_{DR} of connections on a compact Riemann surface X. We will consider a path of connections of the form $(E, \nabla + t\theta)$ which approaches the boundary divisor transversally at the point on the boundary of M_{DR} corresponding to a general Higgs bundle (E, θ). By some meromorphic gauge transformations in § 5 we reduce to the case of a family of connections of the form $d + B + tA$. This is very

2000 ***Mathematics Subject Classification.*** — Primary 34E20; Secondary 14F10, 32G34.
Key words and phrases. — Connection, ODE, Singular perturbation, Turning point, Resurgent function, Laplace transform, Growth rate, Planar tree, Higgs bundle, Moduli space, Compactification, λ-connection, Gauge transformation, Monodromy, Fundamental group, Representation, Iterated integral.

similar to what was treated in [**36**] except that here our matrix B may have poles. We import the vast majority of our techniques directly from there. The difficulty posed by the poles of B is the new phenomenon which is treated here. We are not able to get results as good as the precise asymptotic expansions of [**36**]. We just show in Theorem 6.3 (p. 205) that if $m(t)$ denotes the family of monodromy or transport matrices for a given path, then the Laplace transform $f(\zeta)$ of m has an analytic continuation with locally finite singularities over the complex plane (see Definition 6.2, p. 205). The singularities are what determine the asymptotic behavior of $m(t)$. The upside of this situation is that since we are aiming for less, we can considerably simplify certain parts of the argument. What we don't know is the behavior of $f(\zeta)$ near the singularities: the main question left open is whether f has polynomial growth at the singularities, and if so, to what extent the generalized Laurent series can be calculated from the individual terms in our integral expression for f.

We can get some information about where the singularities are. Fix a general point (E, θ). Recall from [**26, 27, 19, 30**] that the *spectral curve* V is the subset of points in $T^*(X)$ corresponding to eigenforms of θ. We have a proper mapping $\pi : V \to X$. In the case of a general point, V is smooth and the mapping has only simple ramification points. Also there is a tautological one-form

$$\alpha \in H^0(V, \pi^*\Omega^1_X) \subset H^0(V, \Omega^1_V).$$

Finally there is a line bundle L over V such that $E \cong \pi_*(L)$ and θ corresponds to the action of α on the direct image bundle. This is all just a geometric version of the diagonalization of θ considered as a matrix over the function field of X.

Let $\mathcal{R} \subset X$ denote the subset of points over which the spectral curve is ramified, that is the image of the set of branch points of π. It is the set of *turning points* of our singular perturbation problem. Suppose p and q are points in X joined by a path γ. A *piecewise homotopy lifting* of γ to the spectral curve V consists of a collection of paths

$$\widetilde{\gamma} = \{\widetilde{\gamma}_i\}_{i=1,\dots,k}$$

such that each $\widetilde{\gamma}_i$ is a continuous path in V, and such that if we denote by $\gamma_i := \pi \circ \widetilde{\gamma}_i$ the image paths in V, then γ_1 starts at p, γ_k ends at q, and for $i = 1, \dots, k-1$, the endpoint of γ_i is equal to the starting point of γ_{i+1} and this is a point in $\mathcal{R}$. Among these there is a much more natural class of paths which are the *continuous homotopy liftings*, namely those where the endpoint of $\widetilde{\gamma}_i$ is equal to the starting point of $\widetilde{\gamma}_{i+1}$ (which is not necessarily the case for a general piecewise lifting).

Denote by $\Sigma(\gamma) \subset \mathbb{C}$ the set of integrals of the tautological form α along piecewise homotopy liftings of γ, *i.e.* the set of complex numbers of the form

$$\sigma = \int_{\widetilde{\gamma}} \alpha := \sum_{i=1}^{k} \int_{\widetilde{\gamma}_i} \alpha.$$

Let $\Sigma^{\mathrm{cont}}(\gamma)$ be the subset of integrals along the continuous homotopy liftings. The following is the statement of Theorem 6.3 augmented with a little bit of information about where the singularities are.

Theorem 1.1. — *Let p, q be two points on X, and let γ denote a path from p to q. Let $\{(E, \nabla + t\theta)\}$ denote a curve of connections cutting the divisor P_{DR} at a general point (E, θ) and let (V, α, L) denote the spectral data for this Higgs bundle. Let $m(t)$ be the function (with values in $Hom(E_p, E_q)$) whose value at $t \in \mathbb{C}$ is the transport matrix for the connection $\nabla + t\theta$ from p to q along the path γ. Let $f(\zeta)$ denote the Laplace transform of m. Then, f has an analytic continuation with locally finite singularities over the complex plane. The set of singularities which are ever encountered is a subset of the set $\Sigma(\gamma) \subset \mathbb{C}$ of integrals of the tautological form along piecewise homotopy liftings defined above.*

It would have been much nicer to be able to say that the set of singularities is contained in $\Sigma^{\mathrm{cont}}(\gamma)$, however I don't see that this is necessarily the case. However, it might be that the singularities in $\Sigma^{\mathrm{cont}}(\gamma)$ have a special form different from the others. This is an interesting question for further research.

Ths first singularities which are encountered in the analytic continuation of f determine the growth rate of $m(t)$ in a way which we briefly formalize. Suppose that $m(t)$ is an entire function with exponentially bounded growth. We say that $m(t)$ is *rapidly decreasing in a sector*, if for some (open) sector of complex numbers going to ∞, there is $\varepsilon > 0$ giving a bound of the form $|m(t)| \leqslant e^{-\varepsilon|t|}$. Define the *hull* of m by

$$\mathbf{hull}(m) := \{\zeta \in \mathbb{C} \mid e^{-\zeta t} m(t) \text{ not rapidly decreasing in any sector}\}.$$

It is clear from the definition that the set of ζ such that $e^{-\zeta t} m(t)$ is rapidly decreasing in some sector, is open. Therefore $\mathbf{hull}(m)$ is closed. It is also not too hard to see that it is convex (see § 13). Note that the hull is defined entirely in terms of the growth rate of the function m.

Corollary 1.2. — *In the situation of Theorem 1.1, the hull of m is a finite convex polygon with at least two vertices, and all of its vertices are contained in $\Sigma(\gamma)$.*

The above results fall into the realm of *singular perturbation theory* for systems of ordinary differential equations, which goes back at least to Liouville. A steady stream of progress in this theory has led to a vast literature which we don't attempt completely to cover here (and which the reader can explore by using internet and database search techniques, starting for example from the authors mentionned in the bibliography).

Recall that following [**4**], Voros and Ecalle looked at these questions from the viewpoint of "resurgent functions" [**43, 44, 42, 21, 20, 22, 7, 9, 15**]. In the terminology of Ecalle's article in [**7**], the singular perturbation problem we are considering here is an example of *co-equational resurgence*. Our approach is very related to this viewpoint, though self-contained. We use a notion of analytic continuation of the Laplace transform 6.2 which is a sort of weak version of resurgence, like that used in [**15**] and [**9**]. The elements of our expansion 6.1 are what Ecalle calls the "elementary monomials" and the trees which appear in §8 are related to *(co)moulds (co)arborescents*, see [**7**]. Conversion properties related to the trees have been discussed in [**23**] (which is on the subject of KAM theory [**24**]). The relationship with integrals on a spectral curve was explicit in [**14**], [**15**]. The works [**42**], Ecalle's article in [**7**], and [**15**], raise a number of questions about how to prove resurgence for certain classes of singular perturbation problems notably some arising in quantum mechanics. A number of subsequent articles treat these questions; I haven't been able to include everything here but some examples are [**23**], [**16**], [**17**], ... (and apparently [**46**]). In particular [**17**] discuss extensively the way in which the singularities of the Laplace transform determine the asymptotic behavior of the original function, specially in the case of the kinds of integrals which appear as terms in the decomposition 6.1.

There are a number of other currents of thought about the problem of singular perturbations. It is undoubtedly important to pursue the relationship with all of these. For example, the study initiated in [**6**] and continuing with several articles in [**7**], as well as the more modern [**1**] (also Prof. Kawai's talk at this conference) indicates that there is an intricate and fascinating geometry in the propagation of the Stokes phenomenon. And on the other hand it would be good to understand the relationship with the local study of turning points such as in [**8**], [**41**]. The article [**16**] incorporates some aspects of all of these approaches, and one can see [**5**] for a physical perspective. Also works on Painlevé's equations and isomonodromy such as [**11, 28, 34, 45**] are probably relevant .

Even though he doesn't appear in the references of [**36**], the ideas of J.-P. Ramis indirectly had a profound influence on that work (and hence on the present note). This can be traced to at least two inputs as follows:

(1) I had previously followed G. Laumon's course about ℓ-adic Fourier transform, which was partly inspired by the corresponding notions in complex function theory, a subject in which Ramis (and Ecalle, Voros, ...) had a great influence; and

(2) at the time of writing [**36**] I was following N. Katz's course about exponential sums, where again much of the inspiration came from Ramis' work (which Katz mentionned very often) on irregular singularities.

Thus I would like to take this opportunity to thank Jean-Pierre for inspiring such a rich mathematical context.

I would also like to thank the several participants who made interesting remarks and posed interesting questions. In particular F. Pham pointed out that it would be a good idea to look at what the formula for the location of the singularities actually said, leading to the statement of Theorem 6.3 in its above form. I haven't been able to treat other suggestions (D. Sauzin, ...), such as looking at the differential equation satisfied by $f(\zeta)$.