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ON ANALYTIC FAMILIES OF INVARIANT TORI FOR PDES

by

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Dedicated to J.-P. Ramis on the occasion of his 60th birthday

Abstract. — We propose to apply a version of the classical Stokes expansion method to the perturbative construction of invariant tori for PDEs corresponding to solutions quasiperiodic in space and time variables. We argue that, for integrable PDEs all but finite number of the small divisors arising in the perturbative analysis cancel. As an illustrative example we establish such cancellations for the case of KP equation. It is proved that, under mild assumptions about decay of the magnitude of the Fourier modes all analytic families of finite-dimensional invariant tori for KP are given by the Krichever construction in terms of theta-functions of Riemann surfaces. We also present an explicit construction of infinite dimensional real theta-functions and of the corresponding quasiperiodic solutions to KP as sums of an infinite number of interacting plane waves.

***Résumé* (Tores invariants pour certaines EDP).** — Nous proposons d'appliquer la méthode des développements de Stokes à la construction perturbative de tores invariants associés à des solutions d'EDP quasi-périodiques en les variables d'espace et de temps. Pour les EDP intégrables, nous nous intéressons à la compensation de presque tous les petits diviseurs apparaissant dans l'analyse pertubative, *i.e.*, la compensation de tous sauf un nombre fini. Nous traitons de cette compensation en détail sur l'exemple de l'équation KP et nous montrons que dans ce cas, sous des hypothèses faibles portant sur la décroissance de l'amplitude des modes de Fourier, toutes les familles analytiques à tores invariants de dimension finie sont données par la construction de Krichever en termes de fonctions théta de surfaces de Riemann. Nous donnons une construction explicite de fonctions théta réelles de dimension infinie et des solutions de KP quasi-périodiques correspondantes comme somme d'une infinité d'ondes planes en interaction.

1. Introduction

Quasiperiodic solutions of the equations of motion

$$\dot{u} = f(u)$$

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in the form

$$u(t) = U(\phi_1, \dots, \phi_n), \quad \phi_j = \omega_j t + \phi_j^0, \ j = 1, \dots, n$$

for a 2π-periodic in each $\phi_1, \dots, \phi_n$ function U has been studied in the classical mechanics since 19th century. The associated geometric image of linear motion on an n-dimensional torus became widely accepted after creation of KAM theory and of the Arnold-Liouville theory of completely integrable Hamiltonian systems [**2**], although it was already familiar in the physics literature after the A. Einstein's treatment of the Bohr-Sommerfeld quantization rules for integrable systems with many degrees of freedom [**14**]. In particular, the Arnold-Liouville theory applied to a completely integrable Hamiltonian system on a $2n$-dimensional symplectic manifold $u \in M^{2n}$ establishes existence of families of n-dimensional invariant tori depending on n parameters $\boldsymbol{I} = (I_1, \dots, I_n)$

$$u(t \,|\, \boldsymbol{I}) = U(\phi_1, \dots, \phi_n \,|\, \boldsymbol{I}), \quad \phi_j = \omega_j(\boldsymbol{I}) t + \phi_j^0, \ j = 1, \dots, n. \tag{1.1}$$

Changing the values of the action variables $I_1, \dots, I_n$ one represents a $2n$-dimensional domain in the symplectic manifold as a torus fibration. Under the nondegeneracy assumption [**2**] the frequencies $\omega_1(\boldsymbol{I}), \dots, \omega_n(\boldsymbol{I})$ run through all possible directions in an open set. In particular, for generic values of the parameters $\boldsymbol{I}$ the solution (1.1) is a quasiperiodic function in time.

Systems of evolutionary PDEs

$$u_t^a = f^a(u, u_{\boldsymbol{x}}, u_{\boldsymbol{xx}}, \dots), \quad \boldsymbol{x} = (x_1, x_2, \dots, x_d), \quad a = 1, \dots, r \tag{1.2}$$

can be considered as an infinite-dimensional analogue of dynamical systems define on a suitable space of functions of d spatial variables $x_1, \dots, x_d$. Although in certain cases it is possible to develop an infinite-dimensional analogue of the Arnold-Liouville theory of completely integrable Hamiltonian systems and to construct families of infinite-dimensional invariant tori for certain nontrivial examples of nonlinear evolutionary PDEs and, moreover, to develop an infinite-dimensional analogue of KAM theory (see [**28, 7, 24, 35**]) and the related theory of Birkhoff normal forms (see [**15, 21, 22**]), in the most physical applications families of low dimensional invariant tori for PDEs play a prominent role.

For linear PDEs families of one-dimensional invariant tori can be readily found in the form of plane waves

$$u(\boldsymbol{x}, t) = A \cos(k_1 x_1 + \dots + k_d x_d - \omega t + \phi_0). \tag{1.3}$$

The *wave numbers* $k_1, \dots, k_d$ take arbitrary values within some domain of the d-dimensional space, the *frequency*

$$\omega = \omega(k_1, \dots, k_d) \tag{1.4}$$

is determined from the so-called *dispersion relation* substituting the ansatz (1.3) into the equation (1.2). It will be assumed that all branches of the dispersion relation (1.3) are real-valued functions. For any such branch A is a r-component vector determined, in the generic situation, up to a scalar factor called the *amplitude*. The phase shift ϕ_0 can also take an arbitrary value. The solution (1.3) in general is quasiperiodic both in space and time variables. Multidimensional invariant tori for linear PDEs are obtained as linear superpositions of plane waves

$$u(\boldsymbol{x}, t) = \sum_{i=1}^{n} A_i \cos(k_1^i x_1 + \cdots + k_d^i x_d - \omega^i t + \phi_0^i)$$

with arbitrary amplitudes, phases and wave numbers, the frequencies determined as above

$$\omega^i = \omega(k_1^i, \ldots, k_d^i), \quad i = 1, \ldots, n.$$

Note that, in the discussion of invariant tori for PDEs, we will not specify the class of functions[(1)] to be considered.

In many cases families of one-dimensional invariant tori can also be obtained for various nonlinear PDEs as travelling wave solutions

$$u(\boldsymbol{x}, t) = U(\phi \mid \boldsymbol{A}), \quad \phi = k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0. \tag{1.5}$$

Here $U(\phi \mid \boldsymbol{A})$ is a 2π-periodic function in ϕ depending on some number of parameters $\boldsymbol{A} = (A_1, A_2, \ldots)$ that determine the shape of the wave. The wave numbers and phases take arbitrary values. The shape of the wave does not depend on the phase shifts but it may depend on the wave numbers. It is convenient to subdivide the parameters $\boldsymbol{A}$ in two parts

$$\boldsymbol{A} = (k_1, \ldots, k_d; a) \tag{1.6}$$

where the parameter a is a nonlinear analogue of the amplitude. The frequency is to be determined from a nonlinear analogue of the dispersion relation. The latter involves also the amplitude parameters a,

$$\omega = \omega(k_1, \ldots, k_d; a). \tag{1.7}$$

For fixed t the solution (1.5) takes constant values along the hyperplanes

$$k_1 x_1 + \cdots + k_d x_d = \text{const}.$$

The points on the hyperplanes move in the orthogonal directions with the constant phase velocity

$$v = \frac{\omega}{|k|}, \quad |k| = \sqrt{k_1^2 + \cdots + k_d^2}.$$

Example 1.1. — The periodic travelling wave for the Kadomtsev-Petviashvili (KP) equation

$$u_{xt} + \frac{1}{4}(3u^2 + u_{xx})_{xx} + \frac{3}{4}u_{yy} = 0 \tag{1.8}$$

(1) The dynamic on a suitable class of almost periodic functions would probably be the appropriate framework for considering the families of finite-dimensional invariant tori with arbitrary wave numbers.

(here $d = 2$, $x = x_1$, $y = x_2$) can easily be obtained in terms of elliptic functions

(1.9)
$$\begin{aligned} u(x,y,t) &= U(\phi), \quad \phi = kx + ly - \omega t + \phi_0 \\ U(\phi) &= \frac{2k^2}{\pi^2} K^2 \left(\kappa^2 \operatorname{cn}^2 \left[\frac{K}{\pi}\phi; \kappa\right] - \gamma\right) + \frac{c}{6} \\ \omega &= -\frac{c}{4}k + \frac{3}{4}\frac{l^2}{k} - k^3 \frac{K^2}{\pi^2}\left(3\frac{E}{K} + \kappa^2 - 2\right) \\ \gamma &= \frac{E}{K} - 1 + \kappa^2. \end{aligned}$$

Here $\operatorname{cn}[z;\kappa]$ is the Jacobi elliptic function with the modulus $0 \leqslant \kappa \leqslant 1$, $K = K(\kappa)$, $E = E(\kappa)$ are complete elliptic integrals of the first and second kind resp., c is an arbitrary constant.

The functions (1.9) are periodic travelling waves propagating with constant speed in the (x,t)-plane. For $l = 0$ the above formulae reduce to the so-called cnoidal waves for the Korteweg-de Vries (KdV) equation

(1.10)
$$u_t + \frac{1}{4}(3u^2 + u_{xx})_x = 0.$$

The KdV equation is known to arise in a fairly general setting of one-dimensional weakly nonlinear waves with small dispersion (see, *e.g.*, [**33**]). In particular it describes one-dimensional shallow water waves of small amplitude. The y-dependence of solutions to the KP equation (1.8) describes[(2)] slow transversal perturbations of the KdV waves [**23**], [**33**].

The elliptic modulus κ plays the role of the amplitude parameter. At the limiting value $\kappa = 0$ one obtains trivial solution $u = 0$; the frequency takes the value $\omega = -\frac{1}{4}(c\,k + k^3)$. For small positive values of the parameter

$$\varepsilon^2 = k\left[\omega + \frac{1}{4}\left(c\,k + k^3 - 3\frac{l^2}{k}\right)\right] > 0$$

one obtains approximately the plane wave solution

$$u \simeq \frac{c}{6} + A\cos(kx + ly - \omega t + \phi_0), \quad \omega \simeq \frac{1}{4}\left(3\frac{l^2}{k} - c\,k - k^3\right)$$

with the small amplitude

$$A \simeq 2\sqrt{\frac{2}{3}}\varepsilon.$$

More accurate idea about the shape of the solution (1.9) for small amplitudes can be obtained by using *Stokes expansion* method [**37**]; see also Chapter 13 of the Whitham's book [**39**]. We will represent this classical method of the theory of water waves in a

(2)The equation (1.8) is often called KPII to distinguish it from the KPI case. The latter equation differs from (1.8) by the sign in front of the second derivative in y. It also has physical applications but not within the theory of water waves [**23**].