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NALINI JOSHI
KENJI KAJIWARA
MARTA MAZZOCCO

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GENERATING FUNCTION ASSOCIATED WITH THE DETERMINANT FORMULA FOR THE SOLUTIONS OF THE PAINLEVÉ II EQUATION

by

Nalini Joshi, Kenji Kajiwara & Marta Mazzocco

Abstract. — In this paper we consider a Hankel determinant formula for generic solutions of the Painlevé II equation. We show that the generating functions for the entries of the Hankel determinants are related to the asymptotic solution at infinity of the linear problem of which the Painlevé II equation describes the isomonodromic deformations.

Résumé **(Fonction génératrice associée à la formule déterminant pour les solutions de l'équation de Painlevé II)**

On s'intéresse à la formule déterminant de Hankel pour les solutions génériques de l'équation de Painlevé II. On établit une relation reliant les fonctions génératrices des coefficients des déterminants de Hankel aux solutions asymptotiques à l'infini du problème linéaire dont les déformations isomonodromiques sont décrites par cette équation de Painlevé II.

1. Introduction

The Painlevé II equation ($\mathrm{P_{II}}$),

$$\frac{d^2u}{dx^2} = 2u^3 - 4xu + 4\left(\alpha + \frac{1}{2}\right), \tag{1}$$

where α is a parameter, is one of the most important equations in the theory of nonlinear integrable systems. It is well-known that $\mathrm{P_{II}}$ admits unique rational solution when α is a half-integer, and one-parameter family of solutions expressible in terms of the solutions of the Airy equation when α is an integer. Otherwise the solution is non-classical [**13, 14, 17**].

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The rational solutions for $P_{II}(1)$ are expressed as logarithmic derivative of the ratio of certain special polynomials, which are called the "*Yablonski-Vorob'ev polynomials*", [**18, 20**]. Yablonski-Vorob'ev polynomials admit two determinant formulas, namely, Jacobi-Trudi type and Hankel type. The latter is described as follows: For each positive integer N, the unique rational solution for $\alpha = N + 1/2$ is given by

$$u = \frac{d}{dx}\log\frac{\sigma_{N+1}}{\sigma_N},$$

where σ_N is the Hankel determinant

$$\sigma_N = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-2} \end{vmatrix},$$

with $a_n = a_n(x)$ being polynomials defined by the recurrence relation

$$\begin{aligned} &a_0 = x, \qquad a_1 = 1, \\ &a_{n+1} = \frac{da_n}{dx} + \sum_{k=0}^{n-1} a_k\, a_{n-1-k}. \end{aligned} \tag{2}$$

The Jacobi-Trudi type formula implies that the Yablonski-Vorob'ev polynomials are nothing but the specialization of the Schur functions [**12**]. Then, what does the Hankel determinant formula mean? In order to answer this question, a generating function for a_n is constructed in [**6**]:

Theorem 1.1 (**[6]**). — *Let $\theta(x,t)$ be an entire function of two variables defined by*

$$\theta(x,t) = \exp\left(2t^3/3\right)\,\mathrm{Ai}(t^2 - x), \tag{3}$$

where $\mathrm{Ai}(z)$ is the Airy function. Then there exists an asymptotic expansion

$$\frac{\partial}{\partial t}\log\theta(x,t) \sim \sum_{n=0}^{\infty} a_n(x)\,(-2t)^{-n}, \tag{4}$$

as $t \to \infty$ in any proper subsector of the sector $|\arg t| < \pi/2$.

This result is quite suggestive, because it shows that the Airy functions enter twice in the theory of classical solutions of the P_{II}:

(1) in the formula [**3**]

$$u = \frac{d}{dx}\log\mathrm{Ai}\big(2^{1/3}x\big), \qquad \alpha = 0,$$

the one parameter family of classical solutions of P_{II} for integer values of α is expressed by Airy functions,

(2) in formulae (3), (4) the Airy functions generate the entries of determinant formula for the rational solutions.

In this paper we clarify the nature of this phenomenon. First, we reformulate the Hankel determinant formula for generic, namely non-classical, solutions of P_{II} already found in [**10, 11**]. We next construct generating functions for the entries of our Hankel determinant formula. We then show that the generating functions are related to the asymptotic solution at infinity of the isomonodromic problem introduced by Jimbo and Miwa [**9**]. More explicitly, the generating functions we construct are represented formally by series in powers of a variable t that does not appear in the second Painlevé equation. We show that they satisfy two Riccati equations, one in the x variable of P_{II}, the other in the auxiliary variable t. These Riccati equations simultaneously linearise to the two linear systems whose compatibility is given by P_{II}. This is the first time in the literature, to our knowledge, that the construction of the isomonodromic deformation problem has been carried out by starting directly from the Painlevé equation of interest.

This result explains the appearance of the Airy functions in Theorem 1.1. In fact, for rational solutions of P_{II}, the asymptotic solution at infinity of the isomonodromic problem is indeed constructed in terms of Airy functions [**7, 8, 15**].

We expect that the generic solutions of the so-called Painlevé II hierarchy [**1, 2, 4**] should be expressed by some Hankel determinant formula. Of course the generating functions for the entries of Hankel determinant should be related to the asymptotic solution at infinity of the isomonodromic problem for the Painlevé II hierarchy. We also expect that the similar phenomena can be seen for other Painlevé equations. We shall discuss these generalizations in future publications.

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2. Hankel Determinant Formula and Isomonodromic Problem

2.1. Hankel Determinant Formula. — We first prepare the Hankel determinant formula for generic solutions for P_{II} (1). To show the parameter dependence explicitly, we denote equation (1) as $P_{II}[\alpha]$. The formula is based on the fact that the τ functions for P_{II} satisfy the Toda equation,

$$\sigma_n''\sigma_n - (\sigma_n')^2 = \sigma_{n+1}\sigma_{n-1}, \quad n \in \mathbb{Z}, \quad ' = d/dx. \tag{5}$$

Putting $\tau_n = \sigma_n/\sigma_0$ so that the τ function is normalized as $\tau_0 = 1$, equation (5) is rewritten as

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \varphi\psi\tau_n^2, \quad \tau_{-1} = \psi, \quad \tau_0 = 1, \quad \tau_1 = \varphi, \quad n \in \mathbb{Z}. \tag{6}$$

Then it is known that τ_n can be written in terms of Hankel determinant as follows [**11**]:

Proposition 2.1. — *Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be the sequences defined recursively as*

$$(7)\quad a_n = a'_{n-1} + \psi \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} a_i a_j, \quad b_n = b'_{n-1} + \varphi \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} b_i b_j, \quad a_0 = \varphi, \quad b_0 = \psi.$$

For any $N \in \mathbb{Z}$, we define Hankel determinant τ_N by

$$\tau_N = \begin{cases} \det(a_{i+j-2})_{i,j\leqslant N} & N > 0, \\ 1, & N = 0, \\ \det(b_{i+j-2})_{i,j\leqslant |N|} & N < 0. \end{cases} \tag{8}$$

Then τ_N satisfies equation (6).

Since the above formula involves two arbitrary functions φ and ψ, it can be regarded as the determinant formula for general solution of the Toda equation. Imposing appropriate conditions on φ and ψ, we obtain determinant formula for the solutions of $\mathrm{P_{II}}$:

Proposition 2.2. — *Let ψ and φ be functions in x satisfying*

$$\frac{\psi''}{\psi} = \frac{\varphi''}{\varphi} = -2\psi\varphi + 2x, \tag{9}$$

$$\varphi'\psi - \varphi\psi' = 2\alpha. \tag{10}$$

Then we have the following:

(1) $u_0 = (\log\varphi)'$ *satisfies* $P_{\mathrm{II}}[\alpha]$.

(2) $u_{-1} = -(\log\psi)'$ *satisfies* $P_{\mathrm{II}}[\alpha - 1]$.

(3) $u_N = \left(\log\frac{\tau_{N+1}}{\tau_N}\right)'$, *where τ_N is defined by equation (8), satisfies* $P_{\mathrm{II}}[\alpha + N]$.

Proof. — (i) and (ii) can be directly checked by using the relations (9) and (10). Then (iii) is the reformulation of Theorem 4.2 in [**10**]. □

2.2. Riccati Equations for Generating Functions. — Consider the generating functions for the entries as the following formal series

$$F_\infty(x,t) = \sum_{n=0}^{\infty} a_n(x)\ t^{-n}, \quad G_\infty(x,t) = \sum_{n=0}^{\infty} b_n(x)\ t^{-n}. \tag{11}$$

It follows from the recursion relations (7) that the generating functions formally satisfy the Riccati equations. In fact, multiplying the recursion relations (7) by t^{-n} and take the summation from $n = 0$ to ∞, we have:

Proposition 2.3. — *The generating functions $F_\infty(x,t)$ and $G_\infty(x,t)$ formally satisfy the Riccati equations*

$$t\frac{\partial F}{\partial x} = -\psi F^2 + t^2 F - t^2\varphi, \tag{12}$$

$$t\frac{\partial G}{\partial x} = -\varphi G^2 + t^2 G - t^2\psi, \tag{13}$$

respectively.