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by

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Abstract. — We formulate a conjecture which predicts, in many cases, the precise p-adic valuations of the eigenvalues of the Hecke operator T_p acting on spaces of classical modular forms. The conjecture has very concrete consequences in the classical theory, but can also be thought of as saying that there is a lot of unexplained symmetry in many of the Coleman-Mazur eigencurves.

***Résumé* (Questions sur les pentes des formes modulaires). —** Nous formulons une conjecture prédisant, dans de nombreux cas, les valuations p-adiques exactes des valeurs propres de l'opérateur de Hecke T_p agissant sur les espaces de formes modulaires classiques. Cette conjecture a des conséquences très concrètes sur la théorie classique, mais elle suggère aussi de nombreuses symétries inexpliquées concernant les courbes de Coleman-Mazur.

Introduction

Let $N \geqslant 1$ be a fixed integer, and let p denote a fixed prime not dividing N. If $k \in \mathbf{Z}$ then there is a complex vector space $S_k(\Gamma_0(Np))$ of cusp forms of weight k and level Np. This space is finite-dimensional over the complex numbers and comes equipped with an action of the Hecke operator U_p, an endomorphism whose eigenvalues are non-zero complex numbers. The characteristic polynomial of U_p has integer coefficients, which implies that the eigenvalues are algebraic integers. Hence we can consider the eigenvalues as lying in $\mathbf{C}$ or in $\overline{\mathbf{Q}}_l$ for any prime l.

The U_p-eigenvalues fall naturally into two classes, p-old ones and p-new ones. The p-old eigenvalues are the roots of $X^2 - a_pX + p^{k-1}$, where a_p runs through the eigenvalues of T_p acting on $S_k(\Gamma_0(N))$. A deep theorem of Deligne says that the p-old eigenvalues all have complex absolute value $p^{(k-1)/2}$. The p-new eigenvalues are what is left, and it is well-known that these eigenvalues are square roots of p^{k-2}. Hence the complex valuations of these U_p-eigenvalues are known in every case. Moreover, from these definitions it is clear that if $l \neq p$ is a prime then the U_p-eigenvalues are all l-adic units.

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From this point of view, the question that remains about valuations of eigenvalues is:

Question. — *What can one say about the p-adic valuations of the eigenvalues of U_p?*

The term "slopes" is used nowadays to refer to these valuations. A study of the simplest special case, namely $N = 1$ and $p = 2$, shows that the answer is nowhere near as simple as the other cases. The forms which are 2-new at weight k will all have slope $\frac{k-2}{2}$ and this leaves us with the oldforms, whose slopes we can easily compute from the theory of the Newton Polygon, if we know the 2-adic valuations of the eigenvalues of T_2 acting on cusp forms of level 1. The smallest k for which non-zero level 1 cusp forms exist is $k = 12$; the space $S_{12}(\mathrm{SL}_2(\mathbf{Z}))$ is one-dimensional, and T_2 acts as multiplication by -24. Hence the 2-old eigenvalues of U_2 at weight 12 and level 2 are the two roots of $X^2 + 24X + 2^{11}$, and these two roots have 2-adic valuations equal to 3 and 8. Note that $3 \neq 8$, and so the story is already necessarily different to the complex and l-adic cases. We include a short table of valuations and slopes for small weights.

Weight	2-adic valuations of T_2-eigenvalues at level 1	Slopes of U_2 at level 2
12	3	3,8
14		6,6
16	3	3,7,12
18	4	4,8,13
20	3	3,9,9,16
22	5	5,10,10,16
24	3,7	3,7,11,16,20
26	4	4,12,12,12,21

From this table, one wonders whether there is any structure at all in the slopes. However, the purpose of this paper is to suggest that in fact there is a very precise structure here. In fact, in this paper we explain a completely elementary conjectural combinatorial recipe, recursive in the weight k, for generating the above table line by line. In fact, for a large class of pairs (N, p) (including $(1, p)$ for all primes $p < 100$ apart from 59 and 79) we give a conjectural recipe for the valuations of the T_p-eigenvalues at level N, and hence the slopes of U_p at level Np. We strongly believe that there should be a recipe for generating the slopes of U_p at level Np for any N and p, given as an input the slopes for level N and weights at most $p+2$. However we have not yet managed to formulate such a recipe at the present time. In this paper, we offer a recipe only in the case where p is $\Gamma_0(N)$-regular, a term that we shall define later.

Before we explain our conjectural recipe, we shall explain what is known about the slopes of U_p, and what has been conjectured before. The first observation, hinted at by the apparent randomness in the table above, is that to find structure in the slopes of U_p one should, contrary to the complex and l-adic cases, not consider the slopes at one fixed weight, but let the weight vary. There are well-known concrete examples of this phenomenon. For example, a theorem of Hida says that for fixed level, the number of U_p-eigenvalues with slope zero is bounded, and indeed for $k \geqslant 2$ this number depends only on k modulo $p-1$ (resp. modulo 2) for p odd (resp. $p = 2$). As an example of this, we note that there are no slope zero forms in the table above, and we deduce from Hida's theorem that in fact for $N = 1$ and $p = 2$ there will never be any slope zero forms, however high the weight gets.

These theorems about U_p-eigenvalues of slope 0 were generalised by Gouvêa and Mazur to an explicit conjecture in [**11**] about the number of eigenvalues of arbitrary slope as the weight varies. The Gouvêa-Mazur conjecture says that if $M \geqslant 0$ is any integer, then for k and k' sufficiently large (which nowadays means at least $M + 2$) and congruent modulo $(p-1)p^M$, the number of U_p-eigenvalues of slope α at weight k and weight k' should be the same, for any $\alpha \leqslant M$. Experimental evidence for this conjecture was supplied by Mestre in the case where $N = 1$ and p is small. A few years after this conjecture was made, ground-breaking work of Coleman in [**6**] showed that cuspidal eigenforms naturally lay in p-adic analytic families, and an analysis by Wan [**17**] of Coleman's methods showed that one could deduce a weaker version of the Gouvêa-Mazur conjectures, namely that for k and k' sufficiently large, and congruent modulo $(p-1)p^M$, the number of eigenvalues with slope α at these two weights were equal, if $\alpha \leqslant O(\sqrt{M})$. The constants here are all explicit.

Note added in proof: For a few years the gap between the conjecture and the theoretical results was a mystery, but in some sense the mystery was resolved when a counterexample to the Gouvêa-Mazur conjecture was found by the author and F. Calegari in the case $N = 1$ and $p = 59$. This paper was written before the counterexample was found and in fact it was the results in this paper which led the author and Calegari to a study of the particular case $p = 59$, which is the smallest prime for which (at level 1) the results of this paper do not apply. Note that for $N = 1$, although the Gouvêa-Mazur conjecture is false for $p = 59$, it may well still be true for $N = 1$ and $p < 59$, and indeed perhaps the results of this paper are an indication that it is true if p is $\Gamma_0(N)$-regular (see later for the definition). This paper is not about the counterexample at $p = 59$ but about the extra structure discovered for $p < 59$. The counterexample at $p = 59$ is explained in [**4**].

The families in Coleman's work were beautifully interpolated into a mysterious geometric object, constructed by Coleman and Mazur, called an "eigencurve", whose very existence implies deep results about modular forms. One can compute what are essentially local equations for small pieces of these eigencurves for explicit p and N, and computations of this nature have been undertaken by Emerton in [**9**] and Coleman, Stevens and Teitelbaum in [**7**], where for $N = 1$ and $p = 2, 3$ respectively the authors manage to compute the majority of the part of the eigencurve with smallest slope. Computations like this have concrete consequences in the theory—for example, Emerton deduced from his computations that when $N = 1$, the smallest slope of U_2 was periodic as the weight increased, repeating the pattern $3, 6, 3, 4, 3, 5, 3, 4$ indefinitely (one can see the first instance of this pattern in the table above, which already indicates that the table is much too small to be able to indicate what is going on).

The computations of Mestre concerning the Gouvêa-Mazur conjecture were done about ten years ago, and because computers are currently increasing vastly in speed, it was clear that one could go much further nowadays. The author's motivations for actually going further were several—firstly, Wan's results, and unpublished analogous theorems of the author for automorphic forms on definite quaternion algebras, both gave a version of the Gouvêa-Mazur conjecture with $\alpha \leqslant O(\sqrt{M})$ rather than $\alpha \leqslant M$, and this led us to believe that perhaps the Gouvêa-Mazur conjectures were too optimistic. Hence we thought we would make a concerted effort to search for a counterexample (Note added in proof: see [**4**] for the counterexample that we ultimately found). Secondly, several years ago we had come up with an (again unpublished) fast algorithm for computing a matrix representing T_2 on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ and we felt that this would help us with the project. Thirdly, it seemed that a serious computation would be a way to get a "feeling" for the Coleman-Mazur eigencurves. Finally, William Stein has recently written a package that computes spaces of modular forms, and a serious computation seemed like a good way of testing his programs. We should remark that Gouvêa also did many computations since [**11**] was written, and the reader that wants to see the current status of things is strongly recommended to refer to [**10**] or to [**16**].

Our extensive numerical calculations did not (initially) reveal any counterexamples to the Gouvêa-Mazur conjecture (Note added in proof: however they did lead us to the observation that $p = 59$ was somehow different to other primes $p < 59$ and this is what ultimately led to the counterexample). On the contrary, to our surprise, they revealed what in many cases seemed to be far more structure. The Gouvêa-Mazur conjectures predict local constancy of slopes, in some sense, whereas, with the help of the numerical data, we were able to formulate in many cases a new conjecture, which predicted all slopes precisely.

Our investigations of the phenomenon of patterns in slopes were inspired by the aforementioned computations of Emerton, and also by results in Lawren Smithline's 1999 UC Berkeley thesis. We are grateful to both Smithline and Emerton for several helpful remarks. Smithline proves in his thesis that there is some structure to the set of slopes of weight zero 3-adic overconvergent modular forms of level 1, and this structure was one of the reasons why we were inspired to do these computations. We