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NEWTON POLYGONS AND p-DIVISIBLE GROUPS: A CONJECTURE BY GROTHENDIECK

by

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Abstract. — In my talk in 2000 I discussed a conjecture in 1970 by Grothendieck concerning deformations of p-divisible groups; a proof of this conjecture give access to finding properties of Newton polygon strata in the moduli spaces of polarized abelian varieties in positive characteristic.

Résumé **(Polygones de Newton et groupes p-divisibles: une conjecture de Grothendieck)**
En 1970 Grothendieck a formulé une conjecture concernant les déformations de groupes p-divisibles (groupes de Barsotti-Tate). Nous décrivons une démonstration de cette conjecture. Cela donne une information sur des strates définies par le polygone de Newton dans les espaces de modules des variétés abéliennes en caractéristique positive.

Introduction

0.1. We consider p-divisible groups (also called Barsotti-Tate groups) in characteristic p, abelian varieties, their deformations, and we draw some conclusions.

For a p-divisible group (in characteristic p) we can define its Newton polygon. This is invariant under isogeny. For an abelian variety the Newton polygon of its p-divisible group is "symmetric". We are interested in the strata defined by Newton polygons in local deformation spaces, or in the moduli space of polarized abelian varieties.

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Key words and phrases. — Abelian varieties, Newton polygons, p-divisible groups, Barsotti-Tate groups, moduli spaces, Grothendieck conjecture.

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0.2. Grothendieck showed that Newton polygons "go up" under specialization, see [**4**], page 149, see [**11**], Th. 2.3.1 on page 143; we obtain Newton polygon strata as closed subsets in the deformation space of a p-divisible group or in the moduli space of polarized abelian varieties in positive characteristic.

In 1970 Grothendieck conjectured the converse. In [**4**], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: "*... The wishful conjecture I have in mind now is the following: the necessary conditions* [...] *that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular deformation* [...] *we want to know if every sequence of rational numbers satisfying* [...] *these numbers occur as the sequence of slopes of a fiber of G as some point of S.*"

0.3. In this talk we study this conjecture by Grothendieck for p-divisible groups, for abelian varieties, for quasi-polarized p-divisible groups and for polarized abelian varieties. Then we draw conclusions for NP-strata. These results can be found in [**8, 21, 23**].

0.4. We give a proof of this conjecture by Grothendieck. This is done by combining various methods (below we explain the string of ideas leading to this proof); in various stages of the process we need quite different ideas and methods. Hence, in spirit, the proof of a straight statement is not uniform. We have not been able to unify these in one straightforward method. We wonder what Grothendieck would have substituted for our proof.

1. Notations

1.1. We fix some notations. All base fields will be of characteristic $p > 0$. The p-divisible group of an abelian variety X will be denoted by $X[p^\infty]$. We will use *covariant* Dieudonné modules.

We follow [**15**] by writing $G_{m,n}$ for the following p-divisible group (defined over $\mathbb{F}_p$, and considered over every field of characteristic p): this is a p-divisible group of dimension m, with Serre-dual of dimension n; here $m, n \in \mathbb{Z}_{\geqslant 0}$ are coprime integers; we have $G_{1,0} = \mathbb{G}_m[p^\infty]$, and we write $G_{0,1}$ for its Serre dual; for coprime $m, n \in \mathbb{Z}_{>0}$ the formal p-divisible group $G_{m,n}$ is given in the covariant Dieudonné module theory by

$$\mathbb{D}(G_{m,n}) = W[[F,V]]/W[[F,V]] \cdot (F^m - V^n)$$

(in this ring $W[[F,V]]$ we have the relations $FV = p = VF$ and for all $a \in W = W_\infty(K)$, where K is a perfect field, we have $Fa = a^\sigma F$ and $Va^\sigma = aV$; in case $K = \mathbb{F}_p$ this results in a commutative ring). We use $H_{m,n}$ as in [**9**], 5.3; this is a p-divisible group isogenous with $G_{m,n}$; it can be characterized by saying that moreover its endomorphism ring over $\overline{\mathbb{F}_p}$ is the maximal order in its endomorphism algebra.

We need some combinatorial notation concerning Newton polygons:

Throughout the paper we fix a prime number p. We apply notions as defined and used in [**22**], and in [**9**]. For a p-divisible group G, or an abelian variety X, over a field of positive characteristic we use its Newton polygon, abbreviated by NP, denoted by $\mathcal{N}(G)$, respectively $\mathcal{N}(X)$. For dimension d and height $h = d + c$ of G (respectively dimension $g = d = c$ of X) this is a lower convex polygon in $\mathbb{R} \times \mathbb{R}$ starting at $(0, 0)$ ending at (h, c) with integral break points, such that every slope is non-negative and at most equal to one. *We write $\beta \prec \gamma$ if every point of γ is on or below β* (the locus defined by γ contains the one defined by β). For further details we refer to [**22**]. For example, the Newton polygon $\mathcal{N}(G_{m,n})$ consists of $m + n$ slopes equal to $n/(m + n)$. We see that we use the notion of slope as the "V-slope" on p-divisible groups, which amounts to using the "F-slope" on covariant Dieudonné modules.

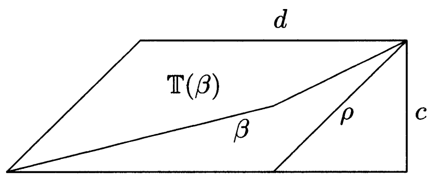
1.2. We use the following notation: we fix integers $h \geqslant d \geqslant 0$, and we write $c := h-d$. We consider Newton polygons ending at (h, c). For a point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we write $(x, y) \prec \gamma$ for the property "*the point (x, y) is on or above the Newton polygon γ*". For a Newton polygon β we write:

$$\mathbb{T}(\beta) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < c,\ y < x,\ (x, y) \prec \beta\},$$

and we define

$$\dim(\beta) := \#(\mathbb{T}(\beta)).$$

Note that for the "ordinary" Newton polygon $\rho := d \cdot (1, 0) + c \cdot (0, 1)$ the set of points $\mathbb{T} = \mathbb{T}(\rho)$ is a parallelogram; this explains our notation. Note that $\#(\mathbb{T}(\rho)) = d \cdot c$.



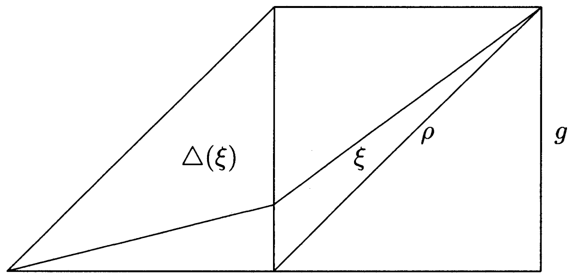
1.3. We fix an integer g. For every *symmetric* Newton polygon ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leqslant g,\ (x, y) \prec \xi\},$$

and we write

$$\operatorname{sdim}(\xi) := \#(\triangle(\xi)).$$

For the ordinary symmetric Newton polygon $\rho = g \cdot ((1, 0) + (0, 1))$ indeed $\triangle = \triangle(\rho)$ is a triangle; this explains our notation. But you can rightfully complain that the "triangle" $\triangle(\xi)$ in general is not a triangle. Note that $\#(\triangle(\rho)) = g(g + 1)/2$.



1.4. A theorem by Grothendieck and Katz, see [**12**], 2.3.2, says that for any family $\mathcal{G} \to S$ of p-divisible groups (in characteristic p) and for any Newton polygon γ there is a unique closed set $W \subset S$ containing all points s at which the fiber has a Newton polygon equal to or lying above γ:

$$s \in W \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathcal{N}(\mathcal{G}_s) \prec \gamma.$$

This set will be denoted by

$$\mathcal{W}_\gamma(\mathcal{G} \to S) \subset S.$$

In case of symmetric Newton polygons we write

$$\mathcal{W}_\gamma(\mathcal{A}_g \otimes \mathbb{F}_p) =: W_\gamma$$

for the Newton polygon stratum given in the moduli space of polarized abelian varieties in characteristic p. We will study this mainly inside $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, the moduli space of *principally* polarized abelian varieties in characteristic p.

1.5. We study formal abelian schemes, and formal p-divisible groups over formal schemes, and we study abelian schemes and p-divisible groups. Without further comments we use the following ideas.

Formal p-divisible groups. — As finite group schemes are "algebraizable", the same holds for certain limits; if $\mathcal{G} \to \mathrm{Spf}(A)$ is a formal p-divisible group, it comes from a p-divisible group over $\mathrm{Spec}(A)$, see [**6**], 2.4.4. We use the passage from formal p-divisible groups over $\mathrm{Spf}(A)$ to p-divisible groups over $\mathrm{Spec}(A)$ without further comments (here A is a complete local ring).

Serre-Tate theory. — Suppose given an abelian variety X_0 over a field K, and its p-divisible group $G_0 := X_0[p^\infty]$. A theorem by Serre and Tate gives an equivalence between formal deformations of (polarized) abelian schemes and the corresponding (quasi-polarized) p-divisible groups, see [**12**], Th. 1.2.1: any formal deformation of G_0 induces uniquely a formal deformation of X_0.