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LESLIE SAPER

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$\mathcal{L}$-MODULES AND THE CONJECTURE OF RAPOPORT AND GORESKY-MACPHERSON

by

Leslie Saper

Abstract. — Consider the middle perversity intersection cohomology groups of various compactifications of a Hermitian locally symmetric space. Rapoport and independently Goresky and MacPherson have conjectured that these groups coincide for the reductive Borel-Serre compactification and the Baily-Borel-Satake compactification. This paper describes the theory of $\mathcal{L}$-modules and how it is used to solve the conjecture. More generally we consider a Satake compactification for which all real boundary components are equal-rank. Details will be given elsewhere [**26**]. As another application of $\mathcal{L}$-modules, we prove a vanishing theorem for the ordinary cohomology of a locally symmetric space. This answers a question raised by Tilouine.

Résumé **($\mathcal{L}$-modules et la Conjecture de Rapoport et Goresky-MacPherson).** — Considérons les groupes de cohomologie d'intersection (de perversité intermédiaire) de diverses compactifications d'un espace localement hermitien symétrique. Rapoport et, indépendamment, Goresky et MacPherson ont conjecturé que ces groupes coïncident pour la compactification de Borel-Serre réductive et la compactification de Baily-Borel-Satake. Cet article décrit la théorie des $\mathcal{L}$-modules et la façon dont elle peut s'employer pour résoudre la conjecture. Plus généralement, nous traitons une compactification de Satake pour laquelle toutes les composantes réelles à la frontière sont de « rang égal ». Les détails en seront disponibles ailleurs [**26**]. Comme application supplémentaire de la théorie des $\mathcal{L}$-modules, nous prouvons un théorème d'annulation sur le groupe de cohomologie ordinaire d'un espace localement symétrique. Ceci répond à une question soulevée par Tilouine.

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1. Introduction

In a letter to Borel in 1986 Rapoport made a conjecture (independently rediscovered by Goresky and MacPherson in 1988) regarding the equality of the intersection cohomology of two compactifications of a locally symmetric variety, the reductive Borel-Serre compactification and the Baily-Borel compactification. In this paper I describe the conjecture, introduce the theory of $\mathcal{L}$-modules which was developed to attack the conjecture, and explain the solution of the conjecture. The theory of $\mathcal{L}$-modules actually applies to the study of many other types of cohomology. As a simple illustration, I will answer at the end of this paper a question raised during the semester by Tilouine regarding the vanishing of the ordinary cohomology of a locally symmetric variety below the middle degree. Except in this final section, proofs are omitted; the details will appear in [**26**].

This paper is an expanded version of lectures I gave during the Automorphic Forms Semester (Spring 2000) at the Centre Émile Borel in Paris; I would like to thank the organizers for inviting me and providing a stimulating environment. During this research I benefited from discussions with numerous people whom I would like thank, in particular A. Borel, R. Bryant, M. Goresky, R. Hain, G. Harder, J.-P. Labesse, J. Tilouine, M. Rapoport, J. Rohlfs, J. Schwermer, and N. Wallach.

2. Compactifications

We consider a connected reductive algebraic group G defined over $\mathbb{Q}$ and its associated symmetric space $D = G(\mathbb{R})/KA_G$, where K is a maximal compact subgroup of $G(\mathbb{R})$ and A_G is the identity component of the $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus in the center of G. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup which for simplicity here we assume to be neat. (Any arithmetic subgroup has a neat subgroup of finite index; the neatness hypothesis ensures that all arithmetic quotients in what follows will be smooth as opposed to V-manifolds or orbifolds.) The locally symmetric space $X = \Gamma\backslash D$ is in general not compact and we are interested in three compactifications (see Figure 1), belonging respectively to the topological, differential geometric, and (if D is Hermitian symmetric) complex analytic categories.

Let $\mathcal{P}$ (resp. $\mathcal{P}_1$) denote the partially ordered set of Γ-conjugacy classes of parabolic (resp. maximal parabolic) $\mathbb{Q}$-subgroups of G. For $P \in \mathcal{P}$, let L_P denote the Levi quotient P/N_P, where N_P is the unipotent radical of P. (When it is convenient we will identify L_P with a subgroup of P via an appropriate lift.) The *Borel-Serre compactification* [**4**] has strata $Y_P = \Gamma_P\backslash P(\mathbb{R})/K_PA_P$ indexed by $P \in \mathcal{P}$ (for $P = G$ we simply have $Y_G = X$). Here $\Gamma_P = \Gamma \cap P$, $K_P = K \cap P$, and A_P is the identity component of the $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus in the center of L_P. The Borel-Serre compactification $\overline{X}$ is a manifold with corners, homotopically equivalent with X itself.

$$
\begin{array}{lcclc}
\text{Borel-Serre} & \overline{X} & = & \coprod_{P\in\mathcal{P}} Y_P, & Y_P = \Gamma_P\backslash P(\mathbb{R})/K_PA_P \\
 & \big\downarrow & & \big\downarrow \text{Collapse } \Gamma_{N_P}\backslash N_P(\mathbb{R}) \text{ fibers} & \\
\text{Reductive Borel-Serre} & \widehat{X} & = & \coprod_{P\in\mathcal{P}} X_P, & X_P = \Gamma_{L_P}\backslash L_P(\mathbb{R})/K_PA_P \\
 & \pi\big\downarrow & & \big\downarrow \text{Project } X_P = X_{P,\ell}\times X_{P,h} \to X_{P,h} = X_{R,h} & \\
\text{Baily-Borel Satake} & X^* & = & \coprod_{R\in\mathcal{P}_1} F_R, & F_R = X_{R,h}
\end{array}
$$

FIGURE 1

The arithmetic subgroup Γ induces arithmetic subgroups $\Gamma_{N_P} = \Gamma\cap N_P$ in N_P and $\Gamma_{L_P} = \Gamma_P/\Gamma_{N_P}$ in L_P. Let $D_P = L_P(\mathbb{R})/K_PA_P$ be the symmetric space associated to L_P and let $X_P = \Gamma_{L_P}\backslash D_P$ be its arithmetic quotient. Each stratum of $\overline{X}$ admits a fibration $Y_P \to X_P$ with fibers being compact nilmanifolds $\Gamma_{N_P}\backslash N_P(\mathbb{R})$. The union $\widehat{X} = \coprod_P X_P$ (with the quotient topology from the natural map $\overline{X} \to \widehat{X}$) is the *reductive Borel-Serre compactification*; it was introduced by Zucker [**34**]. The reductive Borel-Serre compactification is natural from a differential geometric standpoint since the locally symmetric metric on X degenerates precisely along these nilmanifolds near the boundary of $\overline{X}$.

Finally assume now that D is Hermitian symmetric. Then each D_P factors into a product $D_{P,\ell}\times D_{P,h}$, where $D_{P,h}$ is again Hermitian symmetric (see Figure 2). This induces a factorization (modulo a finite quotient) $X_P = X_{P,\ell}\times X_{P,h}$ of the arithmetic quotients and we consider the projection $X_P \to X_{P,h}$ onto the second factor. Now among the different $P\in\mathcal{P}$ that yield the same $X_{P,h}$, let $P^\dagger\in\mathcal{P}_1$ be the maximal one and set $F_{P^\dagger} = X_{P,h}$. Thus each stratum of $\widehat{X}$ has a projection $X_P \to F_{P^\dagger}$. The union $X^* = \coprod_{R\in\mathcal{P}_1} F_R$ (with the quotient topology from the map $\widehat{X}\to X^*$) is the *Baily-Borel-Satake compactification* X^*. Topologically X^* was constructed by Satake [**29**], [**30**] (though the description we have given is due to Zucker [**35**]); if Γ is contained in the group of biholomorphisms of D, the compactification X^* was given the structure of a normal projective algebraic variety by Baily and Borel [**2**].

The simplest example where all three compactifications are distinct is the Hilbert modular surface case. Here $G = R_{k/\mathbb{Q}}\,\mathrm{SL}(2)$ where k is a real quadratic extension. There is only one proper parabolic $\mathbb{Q}$-subgroup P up to $G(\mathbb{Q})$-conjugacy; Y_P is a torus bundle over $X_P = S^1$ and F_P is a point.

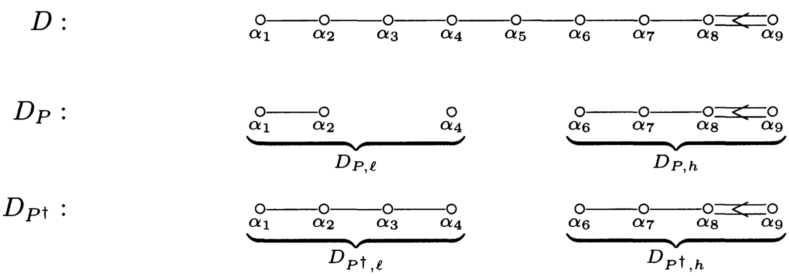

FIGURE 2. An example of $D_P = D_{P,\ell} \times D_{P,h}$ and $D_{P^\dagger} = D_{P^\dagger,\ell} \times D_{P^\dagger,h}$

3. The conjecture

Assume that D is Hermitian symmetric. Let $E \in \mathfrak{Mod}(G)$, the category of finite dimensional regular representations of G and let $\mathbb{E}$ denote the corresponding local system on X. Let $\mathcal{IC}(\widehat{X};\mathbb{E})$ and $\mathcal{IC}(X^*;\mathbb{E})$ denote middle perversity intersection cohomology sheaves[(1)] on $\widehat{X}$ and X^* respectively [**10**].

For example, $\mathcal{IC}(\widehat{X};\mathbb{E}) = \tau^{\leqslant p(\operatorname{codim} X_P)} j_{P*}\mathbb{E}$ if $\widehat{X}$ has only one singular stratum X_P; here j_{P*} denotes the derived direct image functor of the inclusion $j_P : \widehat{X} \setminus X_P \hookrightarrow \widehat{X}$, $\operatorname{codim} X_P$ denotes the topological codimension, $p(k)$ is one of the middle perversities $\lfloor (k-1)/2 \rfloor$ or $\lfloor (k-2)/2 \rfloor$, and $\tau^{\leqslant p(k)}$ truncates link cohomology in degrees $> p(k)$. In general the pattern of pushforward/truncate is repeated over each singular stratum. Note that since $\widehat{X}$ may have odd codimension strata, $\mathcal{IC}(\widehat{X};\mathbb{E})$ depends on the choice of the middle perversity p; on the other hand, since X^* only has even codimension strata, $\mathcal{IC}(X^*;\mathbb{E})$ is independent of p.

***Main Theorem* (Rapoport's Conjecture).** — *Let X be an arithmetic quotient of a Hermitian symmetric space. Then $\pi_*\mathcal{IC}(\widehat{X};\mathbb{E}) \cong \mathcal{IC}(X^*;\mathbb{E})$. (That is, they are isomorphic in the derived category.)*

Following discussions with Kottwitz, Rapoport conjectured the theorem in a letter to Borel [**22**] and later provided motivation for it in an unpublished note [**23**]. Previously Zucker had noticed that the conjecture held for $G = \mathrm{Sp}(4)$, $\mathbb{E} = \mathbb{C}$. The conjecture was later rediscovered by Goresky and MacPherson and described in an unpublished preprint [**11**] in which they also announced the theorem for $G = \mathrm{Sp}(4)$, $\mathrm{Sp}(6)$, and (for $\mathbb{E} = \mathbb{C}$) $\mathrm{Sp}(8)$. The first published appearance of the conjecture was in a revised version of Rapoport's note [**24**] and included an appendix by Saper and Stern giving a proof of the theorem when $\mathbb{Q}\text{-rank}\, G = 1$.

(1) By a "sheaf" we will always mean a complex of sheaves representing an element of the derived category. A derived functor will be denoted by the same symbol as the original functor, thus we will write π_* instead of $R\pi_*$.