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ON THE JACQUET-LANGLANDS CORRESPONDENCE IN THE COHOMOLOGY OF THE LUBIN-TATE DEFORMATION TOWER

by

Matthias Strauch

Abstract. — Let F be a local non-archimedean field, and let $\mathbb{X}$ be a one-dimensional formal $\mathfrak{o}_F$-module over $\overline{\mathbb{F}}_p$ of height n. The formal deformation schemes of $\mathbb{X}$ with Drinfeld level structures give rise to a projective system of rigid-analytic spaces $(M_K)_K$, where K runs through the compact-open subgroups of $G = GL_n(F)$. On the inductive limit H_c^* of the spaces $H_c^*(M_K \otimes \overline{F}^\wedge, \mathbb{Q}_\ell)$ $(\ell \neq p)$ there is a smooth action of $G \times B^\times$, B being a central division algebra over F with invariant $1/n$. For a supercuspidal representation π of G it follows from the work of Boyer resp. Harris-Taylor that in the Grothendieck group of admissible representations of $B^\times$ one has $\mathrm{Hom}_G(H_c^*, \pi) = n \cdot (-1)^{n-1} \mathcal{JL}(\pi)$, $\mathcal{JL}$ denoting the Jacquet-Langlands correspondence. In this paper we propose an approach that is based on a conjectural Lefschetz trace formula for rigid-analytic spaces, and we calculate the contribution coming from the fixed points.

***Résumé* (Sur la correspondance de Jacquet-Langlands dans la cohomologie de la tour de deformations de Lubin-Tate)**

Soient F un corps local non-archimédien et $\mathbb{X}$ un $\mathfrak{o}_F$-module formel de hauteur n sur $\overline{\mathbb{F}}_p$. Les schémas de déformations de $\mathbb{X}$ munies de structures de niveau de Drinfeld fournissent un système projectif d'espaces analytiques rigides $(M_K)_K$, où K parcourt l'ensemble des sous-groupes compacts ouverts de $G = GL_n(F)$. La limite inductive H_c^* des espaces $H_c^*(M_K \otimes \overline{F}^\wedge, \mathbb{Q}_\ell)$ $(\ell \neq p)$ constitue une représentation virtuelle lisse du groupe $G \times B^\times$, B étant une algèbre à division sur F d'invariant $1/n$. Si π est une représentation supercuspidale de G, les travaux de Boyer et Harris-Taylor impliquent que dans le groupe de Grothendieck des représentations admissibles de $B^\times$ on a la relation $\mathrm{Hom}_G(H_c^*, \pi) = n \cdot (-1)^{n-1} \mathcal{JL}(\pi)$, $\mathcal{JL}$ désignant la correspondance de Jacquet-Langlands. Dans cet article nous proposons une approche de ce resultat fondé sur une formule des traces à la Lefschetz conjecturale, et nous calculons la contribution venant des points fixes.

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1. Introduction

Let F be a non-archimedean local field with ring of integers $\mathfrak{o} = \mathfrak{o}_F$. Let $\mathbb{X}$ be a formal $\mathfrak{o}$-module of F-height n over the algebraic closure of the residue field of $\mathfrak{o}$. The functor of deformations of $\mathbb{X}$ is representable by an algebra of formal power series in $n-1$ variables over $\widehat{\mathfrak{o}^{nr}}$. Associated to this algebra there is a rigid-analytic space: the open polydisc of dimension $n-1$. Introducing Drinfeld level structures gives rise to a tower of étale coverings of this space with pro-Galois group $GL_n(\mathfrak{o})$. Moreover, the automorphism group $\mathrm{Aut}_{\mathfrak{o}}(\mathbb{X})$ of $\mathbb{X}$ acts on the deformation space and its coverings, and this action commutes with the action of $GL_n(\mathfrak{o})$. It is convenient to work not only with deformations in the strict sense, *i.e.* ones equipped with an isomorphism of the special fibre to $\mathbb{X}$, but with deformations coming along with a quasi-isogeny of the special fibre to $\mathbb{X}$. In this way one obtains an infinite disjoint union of such towers (indexed by the height of the quasi-isogeny), all being non-canonically isomorphic, and on this tower there is then an action of $GL_n(F) \times B^\times$, where $B = \mathrm{End}_{\mathfrak{o}}(\mathbb{X}) \otimes_{\mathfrak{o}} F$ is a central division algebra over F of dimension n^2. The inductive limit H_c^i of the ℓ-adic étale cohomology groups with compact support of these spaces (after base change to an algebraic closure of F) furnish smooth/continuous representations of $GL_n(F) \times B^\times \times W_F$, where W_F is the Weil group of F, and $\ell \neq p$.

Carayol's conjecture predicts that for a supercuspidal representation π of $GL_n(F)$ the following relation holds true:

$$\mathrm{Hom}_{GL_n(F)}(H_c^{n-1}, \pi) = \mathcal{JL}(\pi) \otimes \sigma(\pi),$$

where $\mathcal{JL}(\pi)$ is the representation of $B^\times$ that is associated to π by the Jacquet-Langlands correspondence, and $\sigma(\pi)$ is, up to twist and dualization, the representation of W_F that is associated to π by the local Langlands correspondence for GL_n. *Cf.* [**Ca1**], sec. 3.3, for a more precise statement also covering the case of non-cuspidal discrete series representations.

In the equal characteristic case, this conjecture has been proven by P. Boyer [**Bo**]. In the mixed characteristic case it may be regarded as being true by the work of M. Harris and R. Taylor [**HT**]. Although they do not state it this way, it seems likely that Carayol's conjecture follows without difficulty from what has been proven in their book, *cf.* [**Ca2**]. Both proofs (equal and mixed characteristic case) use global methods.

In this paper we investigate the alternating sum

$$\mathrm{Hom}_{GL_n(F)}(H_c^*, \pi) := \sum_i (-1)^i \, \mathrm{Hom}_{GL_n(F)}(H_c^i, \pi)$$

as a virtual representation of $B^\times$ by a purely local method. We do not obtain any information about the Weil group representation, except its dimension. Moreover, we pay only attention to the part of the correspondence that concerns the supercuspidal representations. Our approach is based on a *conjectural* Lefschetz trace formula for rigid analytic spaces, and has its origin in Faltings' paper [**Fa**]. Faltings investigated there the corresponding situation of Drinfeld's symmetric spaces and their coverings. In both cases the problem arises that the spaces under consideration are not proper. Hence we cannot expect to express the alternating sum of traces on the cohomology groups as a sum of fixed point mutiplicities. Indeed, simple calculations show that in general there will be an extra term coming from the "boundary". (In the case considered by Faltings however, the "boundary term" turns out to be zero; this is definitively not true in our case, and this is why the situation considered here seems to be more difficult.) In the case $n = 2$ one can use a trace formula for one-dimensional rigid curves proven by R. Huber, *cf.* [**Hu3**]. Huber's trace formula is applied to certain compactifications (in the category of adic spaces) of quasi-compact subspaces, and Huber's trace formula gives an expression of the trace in terms of usual fixed point multiplicities and a contribution from the finitely many compactifying points. While trying to extend Huber's formula to the higher-dimensional case, the author found out that there is another kind of canonical "quasi-compactification" in the category of adic spaces, namely the projective limit of all admissible blow-ups of the corresponding formal schemes representing the deformation functors. The advantage of this compactification is that one has an immediate modular interpretation of the boundary: the boundary has a natural stratification and the geometry and combinatorial structure of the strata can be related to parabolic subgroups of $GL_n(\mathfrak{o}/(\varpi^m))$. Unfortunately, this seems still to be not sufficient to prove that the boundary term (= actual trace minus the number of fixed points) is a sum of parabolically induced virtual characters. We are finally led to consider certain "tubular neighborhoods" of the strata in the boundary. These spaces are insofar interesting as they can be considered as examples of truly non-archimedean spaces over higher-dimensional local fields. But work in this direction has not yet been finished, and hence is not included in this paper. Here we therefore assume that the trace on the alternating sum of the cohomology groups has an appropriate shape, *cf.* sec. 3.5. This conjecture seems to be geometrically justifiable (*cf.* sec. 3.10), and it turns out that the correspondence between the representations of $GL_n(F)$ and $B^\times$ is then given by the number of fixed points (at least as long we consider supercuspidal representations of $GL_n(F)$).

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2. Deformation spaces and their cohomology groups

2.1. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$. Fix a generator ϖ of the maximal ideal of $\mathfrak{o}$, and put $\mathbb{F}_q = \mathfrak{o}/(\varpi)$, q being the cardinality of the residue class field. Moreover, we denote by $\mathbb{F}$ the residue field of the maximal unramified extension $\mathfrak{o}^{nr}$ of $\mathfrak{o}$, and we let $v : F^\times \to \mathbb{Z}$ the valuation determined by $v(\varpi) = 1$.

Let $\mathbb{X}$ be a one-dimensional formal group over $\mathbb{F}$ that is equipped with an action of $\mathfrak{o}$, *i.e.* we assume given a homomorphism $\mathfrak{o} \to \mathrm{End}_{\mathbb{F}}(\mathbb{X})$ such that the action of $\mathfrak{o}$ on the tangent space is given by the reduction map $\mathfrak{o} \to \mathbb{F}_q \subset \mathbb{F}$. Such an object is called a *formal* $\mathfrak{o}$*-module* over $\mathbb{F}$. Moreover, we assume that $\mathbb{X}$ is of F-height n, meaning that the kernel of multiplication by ϖ is a finite group scheme of rank q^n over $\mathbb{F}$.

It is known that for each $n \in \mathbb{Z}_{>0}$ there exists a formal $\mathfrak{o}$-module of F-height n over $\mathbb{F}$, and that it is unique up to isomorphism [**Dr**], Prop. 1.6, 1.7.

Let $\mathcal{C}$ be the category of complete local noetherian $\widehat{\mathfrak{o}^{nr}}$-algebras with residue field $\mathbb{F}$. A *deformation* of $\mathbb{X}$ over an object R of $\mathcal{C}$ is a pair (X, ι), consisting of a formal $\mathfrak{o}$-module X over R which is equipped with an isomorphism $\iota : \mathbb{X} \to X_{\mathbb{F}}$ of formal $\mathfrak{o}$-modules over $\mathbb{F}$, where $X_{\mathbb{F}}$ denotes the reduction of X modulo the maximal ideal $\mathfrak{m}_R$ of R. Sometimes we will omit ι from the notation.

Following Drinfeld [**Dr**], sec. 4B, we define a *structure of level* m on a deformation X over R ($m \geqslant 0$) as an $\mathfrak{o}$-module homomorphism

$$\phi : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \longrightarrow \mathfrak{m}_R,$$

such that $[\varpi]_X(T)$ is divisible by

$$\prod_{a \in (\varpi^{-1}\mathfrak{o}/\mathfrak{o})^n} (T - \phi(a)).$$

Here, $\mathfrak{m}_R$ is given the structure of an $\mathfrak{o}$-module via X, and $[\varpi]_X(T)$ is the power series that gives multiplication by ϖ on X (after having fixed a coordinate T).

For each $m \geqslant 1$ let $K_m = 1 + \varpi^m M_n(\mathfrak{o})$ be the m'th principal congruence subgroup inside $K_0 = GL_n(\mathfrak{o})$. Define the following set-valued functor $\mathcal{M}^{(0)}_{K_m}$ on the category $\mathcal{C}$. For an object R of $\mathcal{C}$ put

$$\mathcal{M}^{(0)}_{K_m}(R) = \{(X, \iota, \phi) \mid (X, \iota) \text{ is a def. over } R,\ \phi \text{ is a level-}m\text{-structure on } X\}/\simeq,$$

where $(X, \iota, \phi) \simeq (X', \iota', \phi')$ iff there is an isomorphism $(X, \iota) \to (X', \iota')$ of formal $\mathfrak{o}$-modules over R, which is compatible with the level structures.