Astérisque

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Astérisque, tome 302 (2005), p. 291-436

<http://www.numdam.org/item?id=AST_2005__302__291_0>

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THE TWISTED WEIGHTED FUNDAMENTAL LEMMA FOR THE TRANSFER OF AUTOMORPHIC FORMS FROM GSp(4) TO GL(4)

by

David Whitehouse

Abstract. — We prove the twisted weighted fundamental lemma for the group $\mathrm{GL}(4) \times \mathrm{GL}(1)$ relative to a certain outer automorphism α, which yields $\mathrm{GSp}(4)$ as a twisted endoscopic group. This version of the fundamental lemma is needed to stabilize the twisted trace formula for the pair $(\mathrm{GL}(4) \times \mathrm{GL}(1), \alpha)$. This stabilized twisted trace formula is required for Arthur's classification of the discrete spectrum of $\mathrm{GSp}(4)$ in terms of automorphic representations of $\mathrm{GL}(4)$.

***Résumé* (Le lemme fondamental tordu pondéré pour le transfert des formes automorphes de GSp(4) à GL(4))**

Nous démontrons le lemme fondamental tordu pondéré pour le groupe $\mathrm{GL}(4) \times \mathrm{GL}(1)$ relativement à un certain automorphisme extérieur α qui permet de décrire $\mathrm{GSp}(4)$ comme groupe endoscopique tordu. Cette version du lemme fondamental est nécessaire pour stabiliser la formule des traces tordue pour le couple $(\mathrm{GL}(4) \times \mathrm{GL}(1), \alpha)$. Cette formule des traces tordues est requise pour la classification d'Arthur du spectre discret de $\mathrm{GSp}(4)$ en termes des représentations automorphes de $\mathrm{GL}(4)$.

1. Introduction

Langlands' functoriality conjecture predicts, in a very precise way, relationships between automorphic representations on different groups. The trace formula is an important tool in proving such relationships. For a reductive group G the trace formula (see [**Art88a**]) gives two expressions for a certain linear form $I(f)$; here f is a suitable function on the adelic points of G. One expression, the geometric side of the trace formula, is given as a sum over conjugacy classes of terms involving orbital integrals, while the other, the spectral side of the trace formula, expresses $I(f)$ in terms associated to the automorphic representations of G. Therefore as the group G is allowed to vary identities between geometric sides produce identities between spectral

2000 ***Mathematics Subject Classification.*** — 11F70, 11F72, 11R39, 20G25, 22E35.
Key words and phrases. — Fundamental lemma, weighted orbital integrals, twisted endoscopy.

sides; out of these one can hope to deduce the relationships between automorphic representations as suggested by Langlands' functoriality conjecture.

Suppose now we are given two groups G_1 and G_2 defined over a number field F. In order to compare the trace formulas for these groups one needs to be able to compare the conjugacy classes in $G_1(F)$ and $G_2(F)$ and to be able to transfer functions from one group to the other given by identities between orbital integrals. In practice, however, for example in the case that G_1 and G_2 are inner forms of each other, one is not quite able to do this. Instead one can only carry out these comparisons over the algebraic closure of F. One is therefore lead to the need for a refinement of this trace formula to a stable trace formula; one in which the geometric side is indexed by stable conjugacy classes and given in terms of stable orbital integrals.

The stabilization of the trace formula was initiated by Langlands in [**Lan83**]. The first problem one encounters is that the distribution $I(f)$ is not stable. In [**Lan83**] Langlands suggested a stabilization of the form

$$I(f) = \sum_H \iota(G,H) S^H(f^H).$$

This sum is over a family of groups $\{H\}$, called elliptic endoscopic groups, attached to G. The distributions S^H are themselves stable and Langlands conjectured a transfer of functions $f \mapsto f^H$ from G to H. One can then hope to compare these stable distributions for different groups. The first case considered was that of SL(2) by Labesse and Langlands in [**LL79**]. For general G, the stabilization of the regular elliptic part of the trace formula was carried out by Langlands in [**Lan83**] under the assumption of a transfer of functions $f \mapsto f^H$. The stabilization of the elliptic singular terms was carried out by Kottwitz in [**Kot86**]. Building on the work of Kottwitz and Langlands, Arthur has now stabilized the full trace formula in a series of papers [**Art02**], [**Art01**] and [**Art03**] under the assumption of certain local conjectures, known as fundamental lemmas, for orbital integrals and weighted orbital integrals; see [**Art02**, Section 5].

The fundamental lemma for orbital integrals has been established in certain cases. It is known for a few groups of low rank, namely for SL(2) by [**LL79**], U(3) by [**Rog90**] and for Sp(4) and GSp(4) by [**Hal97**]. The fundamental lemma has also been established for certain families of groups, for SL(n) by [**Wal91**] and for unitary groups by [**LN04**]. As mentioned in [**Art02**, Section 5] much less is known about the generalization of the fundamental lemma to weighted orbital integrals.

In this paper we are interested in the stabilization of a twisted trace formula. Such a trace formula applies to a group together with an automorphism. The stabilization of the twisted trace formula was begun by Kottwitz and Shelstad in [**KS99**]. For the stabilization of the full twisted trace formula one needs to prove fundamental lemmas for twisted weighted orbital integrals. The statement of the twisted weighted fundamental lemmas is given in the appendix to this paper.

We now turn to the functorial transfer we are concerned with in this paper. We take the group GSp(4) over a number field F. The dual group of GSp(4) is GSp(4, **C**) which has a natural inclusion into GL(4, **C**). Associated to this map of dual groups functoriality suggests a transfer of automorphic representations from GSp(4)/F to GL(4)/F.

There has been much interest in this transfer. Unpublished work of Jacquet, Piatetski-Shapiro and Shalika produced this transfer for generic automorphic representations of GSp(4); this result is proven in [**AS**] using different methods. Results on the transfer from PGSp(4) to PGL(4) have been announced in [**Fli04**]. Flicker uses a special form of the trace formula valid only for certain test functions and so, as mentioned in his paper, the transfer is achieved only for automorphic representations satisfying certain local conditions. The transfer of all automorphic representations from GSp(4) to GL(4) is announced by Arthur in [**Art04**]. In this paper Arthur describes the results of his monograph [**Art**] in the case of GSp(4). The main theorem in [**Art04**] is phrased as a classification theorem for representations of GSp(4). This classification includes a parameterization of the representations of the local groups GSp(4, F_v) into packets together with a decomposition of the discrete spectrum of GSp(4) in terms of automorphic representations of GL(4).

The results of [**Art04**] are achieved by a comparison of the stable trace formula for GSp(4) with a stable twisted trace formula for GL(4) $\times$ GL(1) and a certain automorphism α given in Section 2.4 below. The stabilization of these trace formulas, and hence Arthur's result, is conditional on cases of the fundamental lemma. We now describe which fundamental lemmas are required.

For GSp(4) the fundamental lemma for invariant orbital integrals is proven in [**Hal97**]; see also [**Wei94**]. The weighted fundamental lemma in [**Art02**, Section 5] required for the stabilization of the full trace formula does not apply to GSp(4) since its proper Levi subgroups are products of general linear groups, and therefore do not possess proper elliptic endoscopic groups. Therefore, all the local conjectures required for the stabilization of the trace formula for GSp(4) have been established.

For the stabilization of the twisted trace formula for GL(4) $\times$ GL(1) and the automorphism α, the twisted fundamental lemma for invariant orbital integrals is proven in [**Fli99**]. Flicker's proof is for fields of odd residual characteristic, however, this is sufficient for global applications. A weighted variant of the twisted fundamental lemma, stated in the appendix, is also needed. This is because there are Levi subgroups of GL(4) $\times$ GL(1) that have elliptic twisted endoscopic groups. It is this fundamental lemma which we prove in this paper, we again restrict ourselves to local fields of odd residual characteristic.

The outline of this paper is as follows. We begin in Section 2 by giving some definitions and notations used throughout this paper.

The conjectured twisted weighted fundamental lemma is given by the identity

$$\sum_{k\in\Gamma_{G-\mathrm{reg}}(M(F))} \Delta_{M,K}(\ell',k)r_M^G(k) = \sum_{G'\in\mathcal{E}_{M'}(G)} \iota_{M'}(G,G')s_{M'}^{G'}(\ell').$$

The left hand side consists of a finite linear combination of twisted weighted orbital integrals on the group G^0 with respect to the Levi subset $M = M^0 \rtimes \alpha$. We take M' to be an elliptic twisted endoscopic group for M^0; the right hand side is then a finite linear combination of stable weighted orbital integrals on certain groups G' that contain M' as a Levi subgroup.

From Section 3 onwards we specialize to the twisted weighted fundamental lemma for G^0 equal to $\mathrm{GL}(4)\times\mathrm{GL}(1)$. We begin in Section 3 by determining all endoscopic groups that appear in the statement of the twisted weighted fundamental lemma, and in Section 4 we compute the necessary weight functions, which appear in our weighted orbital integrals.

As above, the twisted weighted fundamental lemma applies to a pair (M, M') of a Levi subset $M = M^0 \rtimes \alpha$ of $G = G^0 \rtimes \alpha$ and an unramified elliptic twisted endoscopic group M' for M^0. When $M^0 = G^0$ we recover the statement of the fundamental lemma proven in [**Fli99**], hence we only consider proper Levi subgroups M^0. There are four pairs (M, M') given in the table below, where E denotes the unramified quadratic extension of the local nonarchimedean field F.

M^0	M'
$(\mathrm{GL}(2)\times\mathrm{GL}(2))\times\mathrm{GL}(1)$	$\mathrm{GL}(2)\times\mathrm{GL}(1)$
$(\mathrm{GL}(1)\times\mathrm{GL}(2)\times\mathrm{GL}(1))\times\mathrm{GL}(1)$	$\mathrm{GL}(2)\times\mathrm{GL}(1)$
$(\mathrm{GL}(1)\times\mathrm{GL}(2)\times\mathrm{GL}(1))\times\mathrm{GL}(1)$	$\mathrm{Res}_{E/F}(\mathrm{GL}(1))\times\mathrm{GL}(1)$
$\mathrm{GL}(1)^4\times\mathrm{GL}(1)$	$\mathrm{GL}(1)^3$

The theorem we prove in this paper is:

Theorem 1.1. — *For each pair (M, M') as above the twisted weighted fundamental lemma is true over local fields of characteristic zero and odd residual characteristic.*

The proof of this theorem is given in Sections 5 through 8. We now outline the proof for each pair. We take F to be a local field of characteristic zero. We let R denote the ring of integers in F. We denote by q the cardinality of the residue field of F that for now we assume is odd and greater than three.

In Section 5 we prove the fundamental lemma for the first pair. We begin by writing both sides of the fundamental lemma in this case as untwisted orbital integrals on $\mathrm{GL}(2,F)$. The identity to be proven then takes the form

$$FL(A): L(A) = R(A)$$

indexed by elements $A\in\mathrm{GL}(2,F)$. Moreover, since both sides vanish if the conjugacy class of A in $\mathrm{GL}(2,F)$ does not intersect $\mathrm{GL}(2,R)$ we may assume that $A\in\mathrm{GL}(2,R)$.