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9. CANONICAL AND QUASI-CANONICAL LIFTINGS IN THE SPLIT CASE

by

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Abstract. — Following Gross we sketch a theory of quasi-canonical liftings when the formal $\mathcal{O}_K$-module of height two and dimension one is replaced by a divisible $\mathcal{O}_K$-module of height one and dimension one in the sense of Drinfel'd.

***Résumé* (Relèvements canoniques et quasi-canoniques dans le cas déployé).** — Suivant Gross, on donne une théorie de relèvements quasi-canoniques dans le cas où le $\mathcal{O}_K$-module de hauteur deux et de dimension un est remplacé par un $\mathcal{O}_K$-module divisible de hauteur un et de dimension un au sens de Drinfel'd.

In this paper, we follow up on a remark by Gross [**G**] and discuss a theory of quasi-canonical liftings when the formal $\mathcal{O}_K$-module of height two and dimension one considered in [**Ww1**] is replaced by a divisible $\mathcal{O}_K$-module of height one and dimension one in the sense of Drinfel'd [**D**]. In this situation the statements analogous to those in [**G**], [**Ww1**] are easy consequences of Lubin-Tate theory and of a slight modification of the Serre-Tate theorem for ordinary elliptic curves, as discussed in the appendix to [**Mes**].

1. Formal moduli of divisible $\mathcal{O}_K$-modules

Let K be a field complete with respect to some discrete valuation. Let $\mathcal{O}_K$ be its ring of integers, $\mathfrak{p} = (\pi)$ its maximal ideal. We assume the residue field $\mathcal{O}_K/\mathfrak{p}$ to be finite and let q denote the number of its elements. For any non-zero ideal $\mathfrak{a} \subset \mathcal{O}_K$ we set $N(\mathfrak{a}) := |\mathcal{O}_K/\mathfrak{a}|$, *i.e.*, $N(\mathfrak{p}^s) = q^s$. Let k be an algebraic closure of $\mathcal{O}_K/\mathfrak{p}$. Let M be the completion of the maximal unramified extension of K in some fixed separable closure K^{sep}. Denote the completion of K^{sep} by C. Let $\mathcal{O}_M$ and $\mathcal{O}_C$ be the rings of integers in M and C respectively.

Following [**D**, §4] a formal group is a group object in the category of formal schemes. For example any group scheme or any discrete group is a formal group in this sense.

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For a formal group F let us denote by F° its connected component. Let $\widehat{\mathcal{C}}$ be the category of complete local noetherian $\mathcal{O}_M$-algebras with residue field k.

Definition 1.1. — Let $R \in \widehat{\mathcal{C}}$. A divisible $\mathcal{O}_K$-module over R is a pair F, where F is a formal group over R and $\gamma_F : \mathcal{O}_K \to \mathrm{End}_R(F)$ is a homomorphism such that F° is a formal $\mathcal{O}_K$-module of height $h < \infty$ in the sense of [**VZ**], and such that

$$F/F^\circ \cong (K/\mathcal{O}_K)^j_{\mathrm{Spf}(R)}$$

for some $j < \infty$. The pair (h, j) will be called type of F.

To ease the notation, we will suppress the structure map γ_F of an $\mathcal{O}_K$-module F and simply write F.

Drinfel'd shows that a divisible $\mathcal{O}_K$-module over k is up to isomorphism given by its type (h, j) (see [**D**, §4]).

Example 1.2. — For $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$ the product group $G = \widehat{\mathbb{G}}_{m,R} \times (\mathbb{Q}_p/\mathbb{Z}_p)_R$ is an example of a divisible module of type $(h, j) = (1, 1)$ over R.

If $R \in \widehat{\mathcal{C}}$ is artinian then the category of fppf-abelian sheaves on R with $\mathcal{O}_K$-structure is an abelian category, the category of $\mathcal{O}_K$-modules over R. It is useful to view the category of divisible $\mathcal{O}_K$-modules over R as a full sub-category of this category.

Definition 1.3. — Fix a divisible $\mathcal{O}_K$-module G over k. A deformation of G to $R \in \widehat{\mathcal{C}}$ is a pair (F, ψ) consisting of a divisible $\mathcal{O}_K$-module F over R together with an isomorphism $\psi : F \otimes_R k \xrightarrow{\cong} G$ of $\mathcal{O}_K$-modules.

The deformations of G to $R \in \widehat{\mathcal{C}}$ form a category in a natural way. One checks that it is a groupoid and moreover that no object of this groupoid has non-trivial automorphisms. The last point is due to the fact that for a deformation F the isomorphism ψ is part of the data. Nevertheless we often omit ψ from the notation.

Definition 1.4. — For any $R \in \widehat{\mathcal{C}}$ let us denote by $\mathcal{D}_G(R)$ the set of isomorphism classes of the groupoid of deformations of G to R. Then $\mathcal{D}_G$ becomes a set-valued functor on $\widehat{\mathcal{C}}$.

Fix a formal $\mathcal{O}_K$-module H_0 of height $h = 1$ over k. It has a trivial deformation space, *i.e.*, $\mathcal{D}_{H_0}(R) = \{\text{point}\}$ for any $R \in \widehat{\mathcal{C}}$. More precisely $\mathcal{D}_{H_0}$ is representable by $\mathcal{O}_M$. This follows easily from the uniqueness of Lubin-Tate modules (see [**Me1**]; see also Remark 1.11(ii) for a far more general result of Drinfel'd). Let us denote by H the unique lift of H_0 to $\mathcal{O}_M$. We assume, as we may, that H is given as the base change

$$H = H_f \otimes_{\mathcal{O}_K} \mathcal{O}_M,$$

where H_f is the Lubin-Tate module over $\mathcal{O}_K$ corresponding to some fixed prime element $\pi \in \mathcal{O}_K$ and some fixed Lubin-Tate series $f \in \mathcal{F}_\pi$. Recall from [**Me1**, Lemma 1.7] that the isomorphism class of H does not depend on these choices. Recall further that for any $R \in \widehat{\mathcal{C}}$ we have $H(R) = \mathfrak{m}_R$ as a set. The $\mathcal{O}_K$-module structure is given as follows: For $q, q' \in H(R)$ and $z \in \mathcal{O}_K$ we have $q +_H q' = H(q, q')$ and $z \cdot_H q = [z]_f(q)$. We often omit the subscript H from the notation.

Now fix some divisible $\mathcal{O}_K$-module G over k of height $h = 1$ such that there is an isomorphism $G/G^\circ \cong (K/\mathcal{O}_K)_k$. Fix an isomorphism of divisible $\mathcal{O}$-modules

$$r \colon G \xrightarrow{\cong} H_0 \times (K/\mathcal{O}_K)_k$$

where H is the unique lift of G° to $\mathcal{O}_M$ as above. Two such isomorphisms differ by an element of the automorphism group of the right hand side. This group is described by the following easy but important lemma.

Lemma 1.5
(1) We have

$$\mathrm{Hom}_{\mathcal{O}_K,k}((K/\mathcal{O}_K)_k, H_0) = \{0\} = \mathrm{Hom}_{\mathcal{O}_K,k}(H_0, (K/\mathcal{O}_K)_k)$$

and

$$\mathrm{End}_{\mathcal{O}_K,k}(H_0) = \mathcal{O}_K = \mathrm{End}_{\mathcal{O}_K,k}((K/\mathcal{O}_K)_k).$$

(2) In particular there is a canonical isomorphism

$$\mathcal{O}_K \times \mathcal{O}_K \longrightarrow \mathrm{End}_{\mathcal{O}_K,k}(H_0 \times (K/\mathcal{O}_K)_k).$$

It induces an isomorphism

$$\mathcal{O}_K^\times \times \mathcal{O}_K^\times \longrightarrow \mathrm{Aut}_{\mathcal{O}_K,k}(H_0 \times (K/\mathcal{O}_K)_k).$$

Proof. — It clearly suffices to prove the first point. We have

$$\mathrm{Hom}_{\mathcal{O}_K,k}((K/\mathcal{O}_K)_k, H_0) = \mathrm{Hom}_{\mathcal{O}_K}(K/\mathcal{O}_K, H_0(k)) = \{0\}$$

by adjunction and because $H_0(k) = \{0\}$. We have

$$\mathrm{Hom}_{\mathcal{O}_K,k}(H_0, (K/\mathcal{O}_K)_k) = \mathrm{Hom}_{\mathcal{O}_K,k}(H_0, (K/\mathcal{O}_K)_k^\circ) = \{0\}$$

because H_0 is connected and $(K/\mathcal{O}_K)^\circ = \{0\}$. We have

$$\mathrm{End}_{\mathcal{O}_K,k}(H_0) = \mathcal{O}_K$$

because by Lubin-Tate theory every endomorphism of H_0 is uniquely given by its differential at zero. We have

$$\mathrm{End}_{\mathcal{O}_K,k}((K/\mathcal{O}_K)_k) = \mathrm{End}_{\mathcal{O}_K}(K/\mathcal{O}_K)$$

by adjunction. Since the natural map

$$\mathcal{O}_K \longrightarrow \mathrm{End}_{\mathcal{O}_K}(K/\mathcal{O}_K)$$

is well known to be an isomorphism we are done. □

We want to sketch a proof of the following theorem (compare the analogous statement in [**VZ**, Theorem 3.8]):

***Theorem 1.6* (Universal deformation).** — *For any $R \in \widehat{\mathcal{C}}$ and fixed isomorphism r there is a natural isomorphism*

$$\eta_R \colon \mathcal{D}_G(R) \xrightarrow{\cong} H(R).$$

In particular $\mathcal{D}_G$ can be given the structure of an $\mathcal{O}_K$-module (depending on r of course). Since we assume $H = H_f \otimes_{\mathcal{O}_K} \mathcal{O}_M$, the $\mathcal{O}_K$-module structure is given by Lubin-Tate theory as recalled above.

The proof will take up the rest of this section. One proceeds as in [**Mes**, appendix]: In the course of the proof we will identify both, $\mathcal{D}_G(R)$ and $H(R)$ for $R \in \widehat{\mathcal{C}}$ artinian, with a certain Ext-group. So let us briefly recall the definition and some basic properties of these groups. A careful discussion can be found in [**Mt**, chapter VII].

For objects M'' and M' of an abelian category $\mathcal{A}$ let

$$\mathcal{E}xt_{\mathcal{A}}(M'', M')$$

denote the groupoid of extensions $(M, p, i) \colon M' \overset{i}{\hookrightarrow} M \overset{p}{\twoheadrightarrow} M''$. It is well known that the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M'', M') & \longrightarrow & \mathrm{Aut}_{\mathcal{E}xt_{\mathcal{A}}(M'',M')}((M,p,i)) \\ \varphi & \longmapsto & \mathrm{id}_M + i \circ \varphi \circ p \end{array}$$

is an isomorphism of groups. In particular the automorphism group of (M, p, i) is trivial if and only if $\mathrm{Hom}_{\mathcal{A}}(M'', M')$ is. Let

$$\mathrm{Ext}_{\mathcal{A}}(M'', M')$$

be the class of isomorphism classes of $\mathcal{E}xt_{\mathcal{A}}(M'', M')$. Assume it to be a set. Sometimes we will not distinguish an extension from its isomorphism class. Using Baer-addition $\mathrm{Ext}_{\mathcal{A}}(M'', M')$ becomes an abelian group in the usual way. For $N' \in \mathcal{A}$ let

$$\delta_{(M,p,i),N'} \colon \mathrm{Hom}_{\mathcal{A}}(M', N') \longrightarrow \mathrm{Ext}_{\mathcal{A}}(M'', N'). \tag{1.1}$$

be the boundary homomorphism.

Apply this in the case that $\mathcal{A}$ is the category of $\mathcal{O}_K$-modules on some fixed artinian $R \in \widehat{\mathcal{C}}$. In this case the Ext-groups are in fact $\mathcal{O}_K$-modules.

Definition 1.7. — Let $R \in \widehat{\mathcal{C}}$ be artinian. For any two $\mathcal{O}_K$-modules M' and M'' over R let

$$\mathrm{Ext}_{\mathcal{O}_K,R}(M'', M')$$

denote the $\mathcal{O}_K$-module of extension classes of M'' by M' constructed above.

Recall that we view the category of divisible $\mathcal{O}_K$-modules on artinian R as a full sub-category of the category of all $\mathcal{O}_K$-modules.

Lemma 1.8 (compare [**Mes**, I.2.4.3]). — *Let $R \in \widehat{\mathcal{C}}$ be artinian. Given an extension of the form*

$$H_R \overset{i}{\hookrightarrow} F \overset{p}{\twoheadrightarrow} (K/\mathcal{O}_K)_R$$

of $\mathcal{O}_K$-modules over R, then F is a divisible $\mathcal{O}_K$-module such that $F^\circ \cong H_R$ and $F/F^\circ \cong (K/\mathcal{O}_K)_R$. If one uses the isomorphism $r\colon G \xrightarrow{\cong} H_0 \times (K/\mathcal{O}_K)_k$ then F becomes a deformation of G to R. This association yields a functor between the groupoid of extensions of $(K/\mathcal{O}_K)_R$ by H_R and the groupoid of deformations of G to R.

Proof. — Since $(K/\mathcal{O}_K)_R$ is totally disconnected and H_R is connected it follows that $i\colon H_R \xrightarrow{\cong} F^\circ$. The snake lemma implies that p induces an isomorphism $p'\colon F/F^\circ \xrightarrow{\cong} (K/\mathcal{O}_K)_R$. It follows that F is divisible. Since $H_R(k) = \{0\}$ the extension $H_R \hookrightarrow F \twoheadrightarrow (K/\mathcal{O}_K)_R$ yields an injective map $F(k) \hookrightarrow (K/\mathcal{O}_K)_R(k) = K/\mathcal{O}_K$. Since k is algebraically closed it is an isomorphism. This isomorphism gives us a canonical splitting map $(K/\mathcal{O}_K)_k \hookrightarrow F \otimes k$. Thus the extension is canonically split over k. Together with the identification $r\colon G \xrightarrow{\cong} H_0 \times (K/\mathcal{O}_K)_k$ we get an isomorphism $\psi\colon F \otimes k \xrightarrow{\cong} G$ such that the pair (F, ψ) is a deformation of G. One checks that it is functorial. □

Proposition 1.9 (compare [**Mes**, appendix Prop.2.1]). — *Assume $R \in \widehat{\mathcal{C}}$ to be artinian. Then the functor of the preceding lemma is an equivalence of groupoids and there is a natural isomorphism*

$$\epsilon_R\colon \mathcal{D}_G(R) \xrightarrow{\cong} \operatorname{Ext}_{\mathcal{O}_K,R}((K/\mathcal{O}_K)_R, H_R).$$

Proof. — fully faithful: It is enough to see that every object in either groupoid has a trivial automorphism group. For deformations, this was noted above. For extensions, recall that the automorphism group is isomorphic to $\operatorname{Hom}_{\mathcal{O}_K,R}((K/\mathcal{O}_K)_R, H_R) = \{0\}$.

essentially surjective: Let F be a deformation of G to R. We need to define homomorphisms $i\colon H_R \hookrightarrow F$ and $p\colon F \twoheadrightarrow (K/\mathcal{O}_K)_R$ such that $p \circ i = 0$. For this we let p on R-valued points be defined as follows :

$$F(R) \twoheadrightarrow F(k) = F \otimes k(k) \xrightarrow[r\circ\psi]{\cong} H_0(k) \times (K/\mathcal{O}_K)_k(k) \underset{\mathrm{pr}_2}{\twoheadrightarrow} K/\mathcal{O}_K = (K/\mathcal{O}_K)_R(R).$$

Since $K/\mathcal{O}_K$ is discrete the kernel of p equals F°. Because R is artinian local it follows that $F^\circ \otimes k = (F \otimes k)^\circ \cong G^\circ \cong H_0$. Since H_R is the unique lift of H_0 to R it follows that F° is isomorphic to H_R and we get the map $i\colon H_R \cong F^\circ \hookrightarrow F$. This proves the first assertion. The second follows by passage to isomorphism classes. □

To calculate the Ext-group, we use

Proposition 1.10. — *For any artinian $R \in \widehat{\mathcal{C}}$ the connecting homomorphism associated to the sequence $\mathcal{O}_K \hookrightarrow K \twoheadrightarrow K/\mathcal{O}_K$ is an isomorphism*

$$\delta_R\colon H(R) = \operatorname{Hom}_{\mathcal{O}_K,R}(\mathcal{O}_K, H_R) \xrightarrow{\cong} \operatorname{Ext}_{\mathcal{O}_K,R}((K/\mathcal{O}_K)_R, H_R).$$