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7. FORMAL MODULI OF FORMAL $\mathcal{O}_K$-MODULES

by

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Abstract. — We define formal $\mathcal{O}_K$-modules and their heights, following Drinfeld. To describe their universal deformations we introduce a formal cohomology group.

***Résumé* (Espaces de modules formels de $\mathcal{O}_K$-modules formels).** — On définit les $\mathcal{O}_K$-modules formels et leurs hauteurs, suivant Drinfeld. Pour décrire leurs déformations universelles, on introduit un groupe de cohomologie formelle.

Notation. — *Except in the proof of Lemma 2.1, all constant coefficients of power series are assumed to be 0.*

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1. Formal modules

Let A, R be commutative rings with 1 and $i : A \to R$ a homomorphism. We also write a instead of $i(a)$ for the image of a under i.

Definition 1.1

1. A *formal A-module over R* is a commutative formal group law $F(X,Y) = X + Y + \cdots \in R[[X,Y]]$ together with a ring homomorphism $\gamma : A \to \operatorname{End}_R(F)$ such that the induced map $A \to \operatorname{End}_R(\operatorname{Lie} F) \cong R$ is equal to the structure map i.
2. For $a \in A$ we write $\gamma(a)(X) = [a]_F(X) = aX + \cdots \in R[[X]]$ for the corresponding endomorphism of F. We will also use the notation $X +_F Y$ instead of $F(X,Y)$.

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3. A *homomorphism of formal A-modules over R* is a homomorphism $\varphi(X)$: $F(X,Y) \to G(X,Y)$ of formal group laws $F(X,Y), G(X,Y)$ over R such that $\varphi \circ \gamma_F(a) = \gamma_G(a) \circ \varphi$ for all $a \in A$. Denote by $\mathrm{Hom}_R(F,G)$ the set of homomorphisms from F to G.

Definition 1.2. — For $r \geq 2$ let $\nu_r = p$, if r is a power of a prime p, and $\nu_r = 1$ else. Denote by

$$C_r(X,Y) = \frac{1}{\nu_r}((X+Y)^r - X^r - Y^r)$$

the *modified binomial form* of degree r.

Consider the functor which assigns to every A-Algebra R the set of formal A-modules over R. It is represented by an algebra Λ_A which is generated by the indeterminate coefficients of the series F and $\gamma(a)$ and whose relations are those which are required by the condition that (F,γ) is a formal module. It has a natural grading: the degree of a coefficient is one less than the degree of the corresponding monomial in X, Y. It is induced by the action of $\mathbb{G}_m$ on $\mathrm{Spf}(A[[t]])$. From this description (or by an elementary calculation) one sees that the grading is compatible with concatenation of power series. The elements of the form ab with $\deg a, \deg b \geq 1$ generate a homogeneous ideal. Let $\tilde{\Lambda}_A$ be the quotient with induced grading $\tilde{\Lambda}_A = \bigoplus \tilde{\Lambda}_A^n$.

Denote by $\mathbb{G}_{a,R}$ the additive formal group law over R. With the canonical R-action $\gamma(a) = aX$, it becomes an R-module over R.

Lemma 1.3. — *If A is an infinite field, then for each formal A-module over A there exists a unique isomorphism with $\mathbb{G}_{a,A}$ whose derivative at zero equals* 1. *In this case there is a canonical isomorphism $\Lambda_A \cong A[c_1, c_2, \ldots]$ as graded algebras where $\deg c_i = i$.*

To prove this lemma, one explicitly computes the desired isomorphism, compare [**D**, Prop. 1.2]. The c_i correspond to the coefficients of a homomorphism to the additive formal group law together with the standard A-module structure.

From now on let K be a complete discretely valued field with finite residue field $\mathbb{F}_q$, where $q = p^l$ for some prime p. Denote by $\mathcal{O}_K$ the ring of integers of K. Let π be a uniformizer.

Theorem 1.4. — *$\Lambda_{\mathcal{O}_K}$ and $\mathcal{O}_K[g_1, g_2, \ldots]$ are non-canonically isomorphic as graded algebras where $\deg g_i = i$.*

Proof. — First we show that $\tilde{\Lambda}_{\mathcal{O}_K}^{n-1} \cong \mathcal{O}_K$ as $\mathcal{O}_K$-modules for all $n \geq 2$. For each i let F_i and $[a]_i$ denote the polynomials of degree i obtained from the universal formal module by leaving out all summands of higher degree. We write

$$F_n(X,Y) = F_{n-1}(X,Y) + \sum_{i=1}^{n-1} c_i X^i Y^{n-i}$$

and

$$[a]_n = [a]_{n-1} + h(a) X^n.$$

Then the c_i and $h(a)$ generate $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. As F is a formal group law, we obtain $\sum_{i=1}^{n-1} c_i X^i Y^{n-i} = \alpha C_n(X,Y)$ (compare [**H**, Lemma 1.6.6]). Note that we need here that we consider elements in $\tilde{\Lambda}_{\mathcal{O}_K}$ and not in $\Lambda_{\mathcal{O}_K}$ itself. In particular, $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$ is generated by α and $h(a)$. The condition that $\gamma : \mathcal{O}_K \to \mathrm{End}(F)$ is a homomorphism implies that modulo $(X,Y)^{n+1}$ we have

$$[ab]_{n-1}(X) + h(ab)X^n = [a]_{n-1}([b]_{n-1}(X) + h(b)X^n) + h(a)(bX)^n,$$

$$F_{n-1}([a]_{n-1}(X) + h(a)X^n, [b]_{n-1}(X) + h(b)X^n) + \alpha C_n(aX, bX) \\ = [a+b]_{n-1}(X) + h(a+b)X^n,$$

and

$$[a]_{n-1}(F_{n-1}(X,Y) + \alpha C_n(X,Y)) + h(a)(X+Y)^n \\ = F_{n-1}([a]_{n-1}(X) + h(a)X^n, [a]_{n-1}(Y) + h(a)Y^n) + \alpha C_n(aX, aY).$$

In $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$ this leads to the relations

$$ah(b) + b^n h(a) = h(ab) \tag{1.1}$$

$$h(a+b) - h(a) - h(b) = \alpha C_n(a,b) \tag{1.2}$$

$$(a^n - a)\alpha = \begin{cases} h(a) & \text{if } n \text{ is not a power of a prime} \\ h(a)p' & \text{if } n = p'^l, \end{cases} \tag{1.3}$$

and these are all relations between the generators α, $h(a)$ of $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. If n is invertible in $\mathcal{O}_K$, then (1.3) shows that each $h(a)$ is a multiple of α. If n is a power of p (where $q = p^l$) but not of q itself, then there exists an $a \in \mathcal{O}_K$ with $a^n - a \notin (\pi)$. From (1.1) we obtain $(a^n - a)h(b) = (b^n - b)h(a)$, thus $h(b)$ is a multiple of $h(a)$. Finally (1.2) shows that α is also a multiple of $h(a)$. Now let n be a power of q. By choosing $h(a) \mapsto (a^n - a)/\pi$ and $\alpha \mapsto p/\pi$ we define an epimorphism of $\mathcal{O}_K$-modules $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K} \to \mathcal{O}_K$. It is well defined as (1.1)-(1.3) are the only relations of $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. It remains to prove that $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$ is generated by $h(\pi)$. Let $M = \tilde{\Lambda}^{n-1}_{\mathcal{O}_K}/(h(\pi))$, and denote by $\overline{x} \in M$ the image of $x \in \tilde{\Lambda}^{n-1}_{\mathcal{O}_K}$. Then (1.1) shows that $\pi\overline{h(b)} = \overline{h(\pi b)} = \pi^n \overline{h(b)}$, thus $\overline{h(\pi b)} = 0$ for all $b \in \mathcal{O}_K$. Besides, (1.3) shows $(\pi^n - \pi)\overline{\alpha} = \overline{h(\pi)p} = 0$, hence $\pi\overline{\alpha} = 0$, and M is an $\mathbb{F}_q$-vector space. As n is a power of q, (1.1) reduces to $a\overline{h(b)} + b\overline{h(a)} = \overline{h(ab)}$. This shows

$$\overline{h(a)} = \overline{h(a^n)} = n\overline{a^{n-1}h(a)} = 0$$

for all a. Then (1.2) implies that $C_n(a,b)\overline{\alpha} = 0$ for all $a, b \in \mathbb{F}_q$. By [**H**, Lemma 21.3.2], there is an $x \in \mathbb{F}_p$ with $C_n(x,1) \neq 0$ in $\mathbb{F}_p$. Thus $\overline{\alpha} = 0$ and $M = 0$.

Hence in all cases $\tilde{\Lambda}^{n-1}_{\mathcal{O}_K} \cong \mathcal{O}_K$, and we have an epimorphism of graded algebras $\mathcal{O}_K[g_1, g_2, \dots] \to \Lambda_{\mathcal{O}_K}$. Here g_i is a lift of a generator of $\tilde{\Lambda}^i_{\mathcal{O}_K}$. The construction of the isomorphism $\Lambda_K \cong K[c_1, c_2, \dots]$ in Lemma 1.3 implies that the canonical morphism $\Lambda_{\mathcal{O}_K} \otimes K \to K[c_1, c_2, \dots]$ which is compatible with the grading is also surjective. Comparing dimensions one sees that the epimorphism $\mathcal{O}_K[g_1, g_2, \dots] \to \Lambda_{\mathcal{O}_K}$ is an isomorphism. □

2. Heights

Let $\mathcal{O}_K$ be as above and let R be a local $\mathcal{O}_K$-algebra of characteristic p with residue field k.

Lemma 2.1. — *Let F, G be formal $\mathcal{O}_K$-modules over R and let $\alpha \in \mathrm{Hom}_R(F, G) \setminus \{0\}$. Then there is a unique integer $h = \mathrm{ht}(\alpha) \geq 0$ and $\beta \in R[[X]]$ with $\alpha(X) = \beta(X^{q^h})$ and $\beta'(0) \neq 0$. The integer h is called the* height $\mathrm{ht}(\alpha)$ of α.

This lemma is analogous to the corresponding result over a field, compare [**H**, 18.3.1]. For $\alpha = 0$ we set $\mathrm{ht}(\alpha) = \infty$.

Proof. — We first show that $\alpha(X) = \beta(X^{p^n})$ for some β with $\beta'(0) \neq 0$. To do this we assume $\alpha(X) \neq 0$ with $(\partial\alpha/\partial X)(0) = 0$ and show that $\alpha(X) = \beta(X^p)$ for some homomorphism β of (not necessarily the same) formal group laws. The claim then follows by induction.

Partial differentiation of $\alpha(F(X, Y)) = G(\alpha(X), \alpha(Y))$ with respect to Y gives

$$\frac{\partial\alpha}{\partial X}(F(X,Y))\frac{\partial F}{\partial Y}(X,Y) = \frac{\partial G}{\partial Y}(\alpha(X), \alpha(Y))\frac{\partial\alpha}{\partial X}(Y).$$

Substituting $Y = 0$ and using $(\partial\alpha/\partial X)(0) = 0$ we obtain

$$\frac{\partial\alpha}{\partial X}(X)\frac{\partial F}{\partial Y}(X,0) = 0.$$

As $(\partial F/\partial Y)(X, 0) = 1 + a_1X + \cdots \in R[[X]]^\times$, we obtain $\frac{\partial\alpha}{\partial X}(X) = 0$. Hence $\alpha(X) = \beta(X^p)$ for some $\beta \in R[[X]]$. Let σ_*F be the formal group law obtained from F by raising each coefficient to the pth power. Then an easy calculation shows that β is a homomorphism from σ_*F to G.

We now have to show that p^n is a power of q. Let $a \in \mathcal{O}_K$. Then

$$[a]_G(\alpha(X)) = \alpha([a]_F(X)) = \beta'(0)i(a)^{p^n}X^{p^n} + \cdots$$

and on the other hand

$$[a]_G(\alpha(X)) = \beta'(0)i(a)X^{p^n} + \cdots.$$

This implies $\beta'(0)(i(a) - i(a^{p^n})) = 0$ with $\beta'(0) \neq 0$, hence $i(a) - i(a^{p^n}) = i(a - a^{p^n})$ maps to 0 in k. Thus $a^{p^n} = a$ for all $a \in \mathbb{F}_q$ and p^n is a power of q. □

Definition 2.2. — The *height* of a formal $\mathcal{O}_K$-module F over R is

$$\mathrm{ht}(F) = \begin{cases} h & \text{if } [\pi]_F \text{ has height } h \\ \infty & \text{if } [\pi]_F = 0. \end{cases}$$

Remark 2.3. — This definition is different from the definition of height of a formal module given in [**H**], where it is defined as the height of the reduction of the module over the residue field.

Lemma 2.4. — *Let R be as above and let (F, γ_F) be the formal $\mathcal{O}_K$-module corresponding to a homomorphism $\varphi : \Lambda_{\mathcal{O}_K} \to R$. Then* $\mathrm{ht}(F) = \min\{i | \varphi(g_{q^i-1}) \neq 0\}$.

Proof. — In the proof of Theorem 1.4 we identified the generator g_{q^i-1} of $\tilde{\Lambda}_{\mathcal{O}_K}^{q^i-1}$ with the coefficient of X^{q^i} of $[\pi](X)$. □

The following lemma reduces the examination of formal modules over fields and of their deformations to formal modules of an especially simple form. For a proof see [**D**, Prop. 1.7].

Lemma 2.5. — *Let (F,γ) be a formal $\mathcal{O}_K$-module of height $h<\infty$ over a separably closed field k of characteristic $p>0$. Then F is isomorphic to a formal module (F',γ') over k with*

$$\begin{array}{rcll} F'(X,Y) & \equiv & X+Y & (\operatorname{mod}\deg q^h),\\ [a]_{F'}(X) & \equiv & aX & (\operatorname{mod}\deg q^h),\\ [\pi]_{F'}(X) & = & X^{q^h}. & \end{array}$$

Such modules are called normal modules.

Fix an integer $h>1$ and let F_0 be a formal $\mathcal{O}_K$-module of height h over k. Assume that R is a local artinian $\mathcal{O}_K$-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Let $I \lhd R$ be an ideal. We set $\overline{R}=R/I$. If F is a lift of F_0 over R, we set $\overline{F}:=F\otimes_R\overline{R}$.

Lemma 2.6. — *Let F,G be lifts of F_0 over R. Then the reduction map*

$$\operatorname{Hom}_R(F,G)\to\operatorname{Hom}_{\overline{R}}(\overline{F},\overline{G}) \tag{2.1}$$

is injective.

Proof. — The reduction map in (2.1) is the composition of finitely many maps

$$\operatorname{Hom}_{R_{n+1}}(F\otimes R_{n+1},G\otimes R_{n+1})\to\operatorname{Hom}_{R_n}(F\otimes R_n,G\otimes R_n),$$

where $R_n=R/I_n$ with $I_n=I\cap\mathfrak{m}^n$. We may therefore assume that $\mathfrak{m}\cdot I=0$. Then I is a finite dimensional k-vector space, and we have $I^2=0$. Let $\alpha(X)=a_1X+a_2X^2+\dots$ be a homomorphism from F to G such that $\alpha(X)\equiv 0 \pmod{I}$. We get

$$\alpha([\pi]_F(X))=[\pi]_G(\alpha(X))=0.$$

Since $\operatorname{ht}(F_0)<\infty$, we have $[\pi]_F(X)\neq 0 \pmod{\mathfrak{m}}$, thus $\alpha=0$ which proves the lemma. □

From now on we may consider $\operatorname{Hom}_R(F,G)$ as a subset of $\operatorname{Hom}_{\overline{R}}(\overline{F},\overline{G})$.

3. Deformations of modules, formal cohomology

Let F be a formal $\mathcal{O}_K$-module of height $h<\infty$ over k, and let M be a finite dimensional k-vector space. A *symmetric 2-cocycle* of F with coefficients in M is a