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16. THE CONNECTION TO EISENSTEIN SERIES

by

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Abstract. — We consider the non-singular Fourier coefficients of the special value of the derivative of a Siegel-Eisenstein series of genus 3 and weight 2. We identify these coefficients with the arithmetic degrees of non-degenerate intersections of arithmetic modular correspondences.

***Résumé* (Relation avec les séries d'Eisenstein).** — Nous identifions les coefficients de Fourier non-dégenerés d'une valeur spéciale de la dérivée d'une série de Siegel-Eisenstein de genre 3 et de poids 2 avec les degrés arithmétiques des intersections de correspondances modulaires arithmétiques.

Introduction

In a previous chapter [**Go2**] an expression was obtained for the arithmetic intersection number of three modular correspondences $(\mathcal{T}_{m_1} \cdot \mathcal{T}_{m_2} \cdot \mathcal{T}_{m_3})$, when their intersection is of dimension 0. This expression is quite complicated, and involves certain local representation densities $\beta_\ell(Q)$ of quadratic forms and a local intersection multiplicity $\alpha_p(Q)$. It is this expression that is the main result of [**GK**]. However, already in the introduction to their paper, Gross and Keating mention that computations of S. Kudla and D. Zagier strongly suggest that the arithmetic intersection number $(\mathcal{T}_{m_1} \cdot \mathcal{T}_{m_2} \cdot \mathcal{T}_{m_3})$ agrees (up to a constant) with a Fourier coefficient of the restriction of the derivative at $s = 0$ of a Siegel-Eisenstein series of genus 3 and weight 2.

In the intervening years since the publication of [**GK**], Kudla has vastly advanced this idea and has in particular proved the analogue of this statement for the intersection of two Hecke correspondences on Shimura curves [**Ku3**]. In fact, Kudla has proposed a whole program which postulates a relation between special values of

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derivatives of Siegel-Eisenstein series and arithmetic intersection numbers of special cycles on Shimura varieties for orthogonal groups, comp. [**Ku4**].

The purpose of the present chapter is to sketch these ideas of Kudla and to derive from Kudla's various papers on the subject the statement alluded to in the introduction of [**GK**]. We stress that what we have done here is simply a task of compilation, since we do not (and cannot) claim to have mastered the automorphic side of the statement in question. We use the results of Katsurada [**Ka**] on local representation densities of quadratic forms, valid even for $p = 2$, to relate the local intersection multiplicities to the derivatives of certain local Whittaker functions, comp. [**Wd2**]. For $p \neq 2$ the corresponding calculations of representation densities are much older and are based on results of Kitaoka [**Kit**].

We thank S. Kudla for his help with this chapter.

1. Decomposition of the intersections of modular correspondences

1.1. To $m \in \mathbb{Z}_{>0}$ we have associated the Deligne-Mumford stack which parametrizes the category of isogenies of degree m between elliptic curves,

$$\mathcal{T}_m(S) = \{\, f\colon E \longrightarrow E' \mid \deg(f) = m \,\}.$$

Here E and E' are elliptic curves over S. Then $\mathcal{T}_m$ maps by a finite unramified morphism to the stack $\mathcal{M}^{(2)} = \mathcal{M} \times_{\operatorname{Spec}\mathbb{Z}} \mathcal{M}$ parametrizing pairs (E, E') of elliptic curves.

Let now $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$ and consider

$$\mathcal{T}(m_1, m_2, m_3) = \{\, \mathbf{f} = (f_1, f_2, f_3) \mid f_i\colon E \longrightarrow E',\ \deg f_i = m_i \,\},$$

the fiber product of $\mathcal{T}_{m_1}, \mathcal{T}_{m_2}, \mathcal{T}_{m_3}$ over $\mathcal{M}^{(2)}$. Denoting by Q the degree quadratic form on $\operatorname{Hom}(E, E')$, we obtain a disjoint sum decomposition,

$$\mathcal{T}(m_1, m_2, m_3) = \coprod_T \mathcal{T}_T. \tag{1.1}$$

Here

$$\mathcal{T}_T(S) = \{\, \mathbf{f} \in \operatorname{Hom}_S(E, E')^3 \mid \frac{1}{2}(\mathbf{f}, \mathbf{f}) = T \,\},$$

where $(\mathbf{f}, \mathbf{f})$ denotes the matrix (a_{ij}) with $a_{ij} = (f_i, f_j) = Q(f_i + f_j) - Q(f_i) - Q(f_j)$. Note that, due to the positive definiteness of Q, the index set in (1.1) is $\operatorname{Sym}_3(\mathbb{Z})^\vee_{\geq 0}$, the set of positive semi-definite half-integral matrices.

Lemma 1.1. — *Let $T \in \operatorname{Sym}_3(\mathbb{Z})^\vee_{>0}$,* i.e., *$T$ is positive definite. Then there exists a unique prime number p such that $\mathcal{T}_T$ is a finite scheme with support lying over the supersingular locus of $\mathcal{M}^{(2)}_p = \mathcal{M}^{(2)} \otimes_{\mathbb{Z}} \mathbb{F}_p$.*

Proof. — Let $(E, E') \in \mathcal{M}^{(2)}$ be in the image of $\mathcal{T}_T$. Since $\operatorname{Hom}(E, E')$ has rank at least 3, it follows that (E, E') has to be a pair of supersingular elliptic curves in some positive characteristic p. To see that p is uniquely determined by T, note that T is represented by the quadratic space over $\mathbb{Q}$ corresponding to the definite quaternion algebra ramified in p. However, by [**Ku3**, Prop. 1.3], there is only one quadratic space with fixed discriminant which represents T. $\square$

1.2. In this chapter we consider, for $T \in \mathrm{Sym}_3(\mathbb{Z})^\vee_{>0}$, the number

$$\widehat{\deg}(\mathcal{T}_T) = \lg(\mathcal{T}_T) \cdot \log p \ ,$$

where p is the unique prime in the statement of Lemma 1.1, and where

$$\lg(\mathcal{T}_T) = \sum_{\xi \in \mathcal{T}_T(\bar{\mathbb{F}}_p)} e_\xi^{-1} \cdot \lg(\mathcal{O}_{\mathcal{T}_T,\xi}),$$

with $e_\xi = |\operatorname{Aut}(\xi)|$. Our aim is to compare $\widehat{\deg}(\mathcal{T}_T)$ with the T^{th} Fourier coefficient of a certain Siegel-Eisenstein series of genus 3 and weight 2.

We first define a class of Eisenstein series, among which will be the one appearing in our main theorem.

2. Eisenstein series and the main theorem

2.1. Let B be a quaternion algebra over $\mathbb{Q}$. We denote by $V = V_B$ the quadratic space defined by B, *i.e.*, B with its norm form Q. We note that the idèle class character usually associated to a quadratic space, $x \mapsto (x, (-1)^{n(n-1)/2}\det(V))_{\mathbb{Q}}$ is in this case the trivial character χ_0 ($4 \mid n$, and $\det(V)$ is a square). Let $H = \mathrm{O}(V)$ be the associated orthogonal group. Let $W = \mathbb{Q}^6$, with standard symplectic form $\langle\ ,\ \rangle$ whose matrix with respect to the standard basis is given by $\left(\begin{smallmatrix} 0 & I_3 \\ -I_3 & 0 \end{smallmatrix}\right)$. Let $G = \mathrm{Sp}(W) = \mathrm{Sp}_6$, and denote by $P = M.N$ the Siegel parabolic subgroup, with

$$M = \{\, m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_3 \,\},$$

$$N = \{\, n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathrm{Sym}_3 \,\}.$$

Let $K = K_\infty.K_f = \prod_v K_v$ be the maximal compact subgroup of $G(\mathbb{A})$ with

$$K_v = \begin{cases} \mathrm{Sp}_6(\mathbb{Z}_p), & \text{if } v = p < \infty; \\ \{\, \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a + ib \in \mathrm{U}_3(\mathbb{R}) \,\}, & \text{if } v = \infty. \end{cases} \tag{2.1}$$

We have the Weil representation ω of $G(\mathbb{A}) \times H(\mathbb{A})$ (for the standard additive character ψ of $\mathbb{A}$ with archimedean component $\psi_\infty(x) = \exp(2\pi i x)$ and of conductor zero at all non-archimedean places) on the Schwartz space $\mathcal{S}(V(\mathbb{A})^3)$ (the action of the elements $P(\mathbb{A}) \times H(\mathbb{A})$ are given by simple formulae [**We**], comp. also (4.1) and (4.2) below). In the local version at a place v, we have a representation ω_v of $G(\mathbb{Q}_v) \times H(\mathbb{Q}_v)$ on $\mathcal{S}(V(\mathbb{Q}_v)^3)$.

We have the Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K = N(\mathbb{A})M(\mathbb{A})K.$$

If $g = nm(a)k \in G(\mathbb{A})$, then

$$|a(g)| = |\det(a)|_{\mathbb{A}}$$

is well-defined. For a character χ of $\mathbb{A}^\times/\mathbb{Q}^\times$, we have the induced representation of $G(\mathbb{A})$, corresponding to $s \in \mathbb{C}$,

$$I(s,\chi) = \{\, \Phi\colon G(\mathbb{A}) \to \mathbb{C}\ K\text{-finite function} \mid \\ \Phi(nm(a)g) = \chi(\det(a)) \cdot |a(g)|^{s+2} \cdot \Phi(g)\,\}.$$

For $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$, we set

$$\Phi(g,s) = (\omega(g)\varphi)(0) \cdot |a(g)|^s.$$

In this way, we obtain an intertwining map

$$\mathcal{S}(V(\mathbb{A})^3) \longrightarrow I(0,\chi_0), \quad \varphi \longmapsto \Phi(g,0)\ . \tag{2.2}$$

Note that $|a(g)|$ is a right K-invariant function on $G(\mathbb{A})$, so $\Phi(g,s)$ is a standard section of the induced representation, *i.e.*, its restriction to K is independent of s. We will also need the local version $I(s,\chi_v)$ of the induced representation at a place v and the $G(\mathbb{Q}_v)$-equivariant map

$$\mathcal{S}(V_v^3) \longrightarrow I(0,\chi_{0,v}). \tag{2.3}$$

2.2. Returning to the global situation, we consider the Eisenstein series associated to $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$,

$$E(g,s,\Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s).$$

This series is absolutely convergent for $\mathrm{Re}(s) > 2$, and defines an automorphic form. It has a meromorphic continuation and a functional equation with $s = 0$ as its center of symmetry.

We will now make a specific choice of Φ which will define an *incoherent* Eisenstein series. Let $B = \mathrm{M}_2(\mathbb{Q})$ and let $V(\mathbb{Z}_p) = \mathrm{M}_2(\mathbb{Z}_p)$ for any p. We let $\varphi_f = \otimes\varphi_p = \otimes\,\mathrm{char}\ V(\mathbb{Z}_p)$, and let $\Phi_f = \otimes\Phi_p$ be the corresponding factorizable standard section. For Φ_∞ we take the standard section uniquely determined by

$$\Phi_\infty(k,0) = \det(\underline{k})^2,$$

where $k \in K_\infty$ is the image of $\underline{k} \in \mathrm{U}_3(\mathbb{R})$ under the natural identification in (2.1). Then by [**Ku3**, (7.13)], Φ_∞ is the image of the Gaussian φ_∞ under the local map (2.3) for $v = \infty$, where the local quadratic space is V_∞^+, the positive-definite quadratic space corresponding to the Hamilton quaternion algebra over $\mathbb{R}$, and where

$$\varphi_\infty(x) = \exp(-\pi\ \mathrm{tr}(x,x)), \quad x \in (V_\infty^+)^3. \tag{2.4}$$

Since $V_\infty^+ \otimes V(\mathbb{A}_f)$ does not correspond to a quaternion algebra over $\mathbb{Q}$, the standard section $\Phi = \Phi_\infty \otimes \Phi_f$ is *incoherent* in the sense of *loc. cit.*, and hence (*loc. cit.*, Theorem 2.2),

$$E(g, 0, \Phi) \equiv 0.$$

Consider the Fourier expansion of $E(g, s, \Phi)$,

$$E(g, s, \Phi) = \sum_{T \in \mathrm{Sym}_3(\mathbb{Q})} E_T(g, s, \Phi),$$

where

$$E_T(g, s, \Phi) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(ng, s, \Phi) \cdot \psi_T(n)^{-1} dn,$$

with

$$\psi_T(n(b)) = \psi(\mathrm{tr}(Tb)), \quad b \in \mathrm{Sym}_3(\mathbb{A}). \tag{2.5}$$

For $T \in \mathrm{Sym}_3(\mathbb{Q})$ with $\det(T) \neq 0$, the Fourier coefficient has an explicit expression as a product

$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v), \tag{2.6}$$

see [**Ku3**, (4.4)]. Here $W_{T,v}(g_v, s, \Phi_v)$ is the local Whittaker function, cf. section 5. The local Whittaker functions are entire (cf. [**Ku3**, (4.2) and (4.3)]), and the product (2.6) is absolutely convergent and holomorphic in $s = 0$. More precisely, for $\mathrm{Re}(s) > 2$, the identity (2.6) holds and for almost all places p, the local factor at p on the right hand side equals $\zeta_p(s+2)^{-1} \cdot \zeta_p(2s+2)^{-1} = (1 - p^{-s-2}) \cdot (1 - p^{-2s-2})$, and for all places the local factor is an entire function.

2.3. For $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$, let

$$\mathrm{Diff}(T, V) = \{\, p \mid T \text{ not represented by } V(\mathbb{Q}_p) \,\}.$$

Then the cardinality $|\mathrm{Diff}(T, V)|$ is odd, cf. [**Ku3**, Cor. 5.2]. Moreover we have $W_{T,p}(g_p, 0, \Phi_p) \equiv 0$ for $p \in \mathrm{Diff}(T, V)$, cf. [**Ku3**, Prop. 1.4]. On the other hand, $W_{T,\infty}(g_\infty, 0, \Phi_\infty) \neq 0$, cf. [**Ku3**, Prop. 9.5]. Hence

$$\operatorname*{ord}_{s=0} E_T(g, s, \Phi) \geq |\mathrm{Diff}(T, V)|.$$

In particular, if $E'_T(g, 0, \Phi) \neq 0$, then $\mathrm{Diff}(T, V) = \{p\}$ for a unique prime number p.

2.4. We may now formulate our main theorem.

Theorem 2.1. — *Let* $V = \mathrm{M}_2(\mathbb{Q})$ *and let* $\Phi = \Phi_\infty \otimes \Phi_f$ *be the incoherent standard section as above. Let* $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$ *with* $\mathrm{Diff}(T, V) = \{p\}$.

(i) *If* $T \notin \mathrm{Sym}_3(\mathbb{Z})^\vee$, *then* $\mathcal{T}_T = \varnothing$ *and* $\widehat{\deg}(\mathcal{T}_T) = 0$ *and* $E'_T(g, 0, \Phi) \equiv 0$.