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FABRIZIO ANDREATTA

ADRIAN IOVITA

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GLOBAL APPLICATIONS OF RELATIVE (φ, Γ)-MODULES I

by

Fabrizio Andreatta & Adrian Iovita

Abstract. — In this paper, given a smooth proper scheme X over a p-adic DVR and a p-power torsion étale local system $\mathbb{L}$ on it, we study a family of sheaves associated to the cohomology of local, relative (φ, Γ)-modules of $\mathbb{L}$ and their cohomology. As applications we derive descriptions of the étale cohomology groups on the geometric generic fiber of X with values in $\mathbb{L}$, as well as of their classical (φ, Γ)-modules, in terms of cohomology of the above mentioned sheaves.

***Résumé* (Applications globales des (φ, Γ)-modules relatifs I).** — Étant donné un schéma propre et lisse X défini sur un anneau de valuation discrète et un système local $\mathbb{L}$, étale, de torsion sur X on étudie une famille de faisceaux associés à la cohomologie des (φ, Γ)-modules locaux relatifs de $\mathbb{L}$ et leur cohomologie. Comme application on déduit une description des groupes de cohomologie étales sur la fibre générique géométrique de X à valeurs dans $\mathbb{L}$, et de leurs (φ, Γ)-modules classiques en termes de la cohomologie des faisceaux mentionnés plus haut.

1. Introduction

Let p be a prime integer, K a finite extension of $\mathbf{Q}_p$ and V its ring of integers. In [**15**], J.-M. Fontaine introduced the notion of (φ, Γ)-modules designed to classify p-adic representations of the absolute Galois group G_V of K in terms of semi-linear data. More precisely, if T is a p-adic representation of G_V, i.e. T is a finitely generated $\mathbf{Z}_p$-module (respectively a $\mathbf{Q}_p$-vector space of finite dimension) with a continuous action of G_V, one associates to it a (φ, Γ)-module, denoted $\mathrm{D}_V(T)$. This is a finitely generated module over a local ring of dimension two $\mathbf{A}_V$ (respectively a finitely generated free module over $\mathbf{B}_V := \mathbf{A}_V \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$) endowed with a semi-linear Frobenius endomorphism φ and a commuting, continuous, semi-linear action of the group $\Gamma_V := \mathrm{Gal}(K(\mu_{p^\infty})/K)$ such that $(\mathrm{D}_V(T), \varphi)$ is étale. This construction makes

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the group whose representations we wish to study simpler with the drawback of making the coefficients more complicated. It could be seen as a weak arithmetic analogue of the Riemann-Hilbert correspondence between representations of the fundamental group of a complex manifold and vector bundles with integrable connections. The main point of this construction is that one may recover T with its G_V-action directly from $\mathrm{D}_V(T)$ and, therefore, all the invariants which can be constructed from T can be described, more or less explicitly, in terms of $\mathrm{D}_V(T)$. For example

(*) one can express in terms of $\mathrm{D}_V(T)$ the Galois cohomology groups $\mathrm{H}^i(K,T) = \mathrm{H}^i(\mathrm{G}_V,T)$ of T.

More precisely, let us choose a topological generator γ of Γ_V and consider the complex

$$\mathscr{C}^\bullet(T) : \mathrm{D}_V(T) \xrightarrow{d_0} \mathrm{D}_V(T) \oplus \mathrm{D}_V(T) \xrightarrow{d_1} \mathrm{D}_V(T)$$

where $d_0(x) = \big((1-\varphi)(x), (1-\gamma)(x)\big)$ and $d_1(a,b) = (1-\gamma)(a) - (1-\varphi)(b)$. It is proven in [**18**] that for each $i \geq 0$ there is a natural, functorial isomorphism

$$\mathrm{H}^i(\mathscr{C}^\bullet(T)) \cong \mathrm{H}^i(\mathrm{G}_V, T).$$

Moreover, for $i = 1$ this isomorphism was made explicit in [**9**]: let (x,y) be a 1-cocycle for the complex $\mathscr{C}^\bullet(T)$ and choose $b \in \mathbf{A} \otimes_{\mathbf{Z}_p} T$ such that $(\varphi - 1)(b) = x$. Define the map $C_{(x,y)} \colon \mathrm{G}_V \longrightarrow \mathbf{A} \otimes_{\mathbf{Z}_p} T$ by

$$C_{(x,y)}(\sigma) = (\sigma' - 1)/(\gamma - 1)y - (\sigma - 1)b,$$

where σ' is the image of σ in Γ_V. One can prove that the image of $C_{(x,y)}$ is in fact contained in T, that $C_{(x,y)}$ is a 1-cocycle whose cohomology class $[C_{(x,y)}] \in \mathrm{H}^1(\mathrm{G}_V,T)$ only depends on the cohomology class $[(x,y)] \in \mathrm{H}^1(\mathscr{C}^\bullet(T))$. Moreover, the isomorphism $\mathrm{H}^1(\mathscr{C}^\bullet(T)) \cong \mathrm{H}^1(\mathrm{G}_V,T)$ above is then defined by $[(x,y)] \mapsto [C_{(x,y)}]$.

As a consequence of (*) we have explicit descriptions of the exponential map of Perrin-Riou (or more precisely its "inverse" (see [**15**], [**7**], [**9**]) and an explicit relationship with the "other world" of Fontaine's modules: $\mathrm{D}_{\mathrm{dR}}(T), \mathrm{D}_{\mathrm{st}}(T), \mathrm{D}_{\mathrm{cris}}(T)$ (see [**9**], [**5**]).

Despite being a very useful tool, in fact the only one which allows the general classification of integral and torsion p-adic representations of G_V, the (φ,Γ)-modules have an unpleasant limitation. Namely, $\mathrm{D}_V(T)$ could not so far be directly related to geometry when T is the G_V-representation on a p-adic étale cohomology group (over $\overline{K}$) of some smooth proper algebraic variety defined over K. Here is a relevant passage from the Introduction to [**15**]: "Il est clair que ces constructions sont des cas particuliers de constructions beaucoup plus générales. On doit pouvoir remplacer les corps que l'on considère ici par des corps des fonctions de plusieurs variables ou certaines de leurs complétions. En particulier (i) la loi de réciprocité explicite énoncée au no. 2.4 doit se généraliser et éclairer d'un jour nouveau les travaux de Kato sur ce sujet; (ii) ces constructions doivent se faisceautiser et peut être donner une approche nouvelle des théorèmes de comparaison entre les cohomologies p-adiques."

The first part of the program sketched above, i.e. the construction of relative (φ,Γ)-modules was successfully carried out in [**1**]. The main purpose of the present article is to continue Fontaine's program. In particular various relative analogues, local and global, of $(*)$ are proven.

Let us first point out that in the relative situation, over a "small"-V-algebra R (see §2) there are several variants of (φ,Γ)-module functors, denoted $\mathfrak{D}_R(-)$ (arithmetic), $\mathrm{D}_R(-)$ (geometric), $\widetilde{\mathfrak{D}}_R(-)$ (tilde-arithmetic), $\widetilde{\mathrm{D}}_R(-)$ (tilde-geometric) and their overconvergent counterparts $\mathfrak{D}^\dagger_R(-)$, $\mathrm{D}^\dagger_R(-)$, $\widetilde{\mathfrak{D}}^\dagger_R(-)$ and $\widetilde{\mathrm{D}}^\dagger_R(-)$. For simplicity of exposition let us explain our results in terms of $\mathfrak{D}_R(-)$ and $\widetilde{\mathfrak{D}}_R(-)$.

I) *Local results.* This is carried on in §3 together with the appendices §A and §B. Let R be a "small" V-algebra. Fix an algebraic closure Ω of the fraction field of R and let η be the associated geometric generic point of $\mathrm{Spec}(R)$. Denote by $\overline{R}$ the union of all normal finite extensions of R contained in Ω, which are étale R-algebras after inverting p. Let M be a finitely generated $\mathbf{Z}_p$-module with continuous action of $\mathscr{G}_R := \pi_1^{\mathrm{alg}}\big(\mathrm{Spm}(R_K,\eta)\big)$ and let $\mathrm{D} := \widetilde{\mathfrak{D}}_R(M)$. Then D is a finitely generated $\widetilde{\mathbf{A}}_{\overline{R}}$-module endowed with commuting actions of a semi-linear Frobenius φ and a linear action of the group Γ_R (see §2.) As in the classical case, Γ_R is a much smaller group that $\mathscr{G}_R$. It is the semidirect product of Γ_V and of a group isomorphic to $\mathbf{Z}_p^d$ where d is the relative dimension of R over V.

Let $\mathscr{C}^\bullet(\Gamma_R,\mathrm{D})$ be the standard complex of continuous cochains computing the continuous Γ_R-cohomology of D and denote $\mathscr{T}_R^\bullet(\mathrm{D})$ the mapping cone complex of the morphism $(\varphi-1)\colon \mathscr{C}^\bullet(\Gamma_R,\mathrm{D}) \longrightarrow \mathscr{C}^\bullet(\Gamma_R,\mathrm{D})$. Then, Theorem 3.2 states that we have natural isomorphisms, functorial in R and M,

$$\mathrm{H}^i_{\mathrm{cont}}(\mathscr{G}_R, M) \cong \mathrm{H}^i\big(\mathscr{T}_R^\bullet(\mathrm{D})\big) \quad \text{for all } i \geq 0.$$

The maps are defined in §3 in an explicit way, following Colmez's description in the classical case. The input of Fontaine's construction of the classical (φ,Γ)-modules was to replace modules over perfect, non-noetherian rings with modules over smaller rings: "C'est d'ailleurs [...] que j'ai compris l'intérêt qu'il avait à ne pas remplacer $k(\!(\pi)\!)$ par sa clôture radicielle" Indeed, "[...]ceci permet d'introduire des techniques différentielles". Motivated by the same needs, in view of applications to comparison isomorphisms, we show in appendix §A that one can replace the module $\widetilde{\mathfrak{D}}_R(M)$ over the ring $\widetilde{\mathbf{A}}_{\overline{R}}$, which is not noetherian, with the smaller (φ,Γ_R)-module $\mathfrak{D}_R(M) \subset \widetilde{\mathfrak{D}}_R(M)$ over the noetherian, regular domain $\mathbf{A}_R$ of dimension $d+1$. We show that the natural map

$$\mathrm{H}^i_{\mathrm{cont}}(\Gamma_R, \mathfrak{D}_R(M)) \longrightarrow \mathrm{H}^i_{\mathrm{cont}}(\Gamma_R, \widetilde{\mathfrak{D}}_R(M)) \quad \text{for all } i \geq 0$$

is an isomorphism. The proof follows and slightly generalizes the Tate-Sen method in [**2**]. In particular, one has a natural isomorphism

$$\mathrm{H}^i_{\mathrm{cont}}(\mathscr{G}_R, M) \cong \mathrm{H}^i\big(\mathscr{T}_R^\bullet(\mathfrak{D}_R(M))\big) \quad \text{for all } i \geq 0,$$

where $\mathscr{T}_R^\bullet(\mathfrak{D}_R(M))$ is the mapping cone complex of the map

$$(\varphi-1)\colon \mathscr{C}^\bullet(\Gamma_R, \mathfrak{D}_R(M)) \to \mathscr{C}^\bullet(\Gamma_R, \mathfrak{D}_R(M)).$$

II) *Global results.* This is carried on in §4, §5, §6. The setting for §4 and §5 is the following. Let X be a smooth, proper, geometrically irreducible scheme of finite type over V and let $\mathbb{L}$ denote a locally constant étale sheaf of $\mathbf{Z}/p^s\mathbf{Z}$-modules (for some $s \geq 1$) on the generic fiber X_K of X. Let $\mathscr{X}$ denote the formal completion of X along its special fiber and let X_K^{rig} be the rigid analytic space attached to X_K. Fix a geometric generic point $\eta = \mathrm{Spm}(\mathbb{C}_{\mathscr{X}})$ and set $\mathbf{L}$ the fiber of $\mathbb{L}$ at η.

To each $\mathscr{U} \to \mathscr{X}$ étale such that $\mathscr{U}$ is affine, $\mathscr{U} = \mathrm{Spf}(R_{\mathscr{U}})$, with $R_{\mathscr{U}}$ a small V-algebra and a choice of local parameters $(T_1, T_2, \ldots, T_d)$ of $R_{\mathscr{U}}$ (as in §2) we attach the relative (φ, Γ)-module $\widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L}) := \widetilde{\mathfrak{D}}_{R_{\mathscr{U}}}(\mathbf{L})$. However, the association $\mathscr{U} \longrightarrow \widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L})$ is not functorial because of the dependence of $\widetilde{\mathfrak{D}}_{\mathscr{U}}(\mathbf{L})$ on the choice of the local parameters. In other words the relative (φ, Γ)-module construction does not sheafify.

Nevertheless due to I) above, the association $\mathscr{U} \longrightarrow \mathrm{H}^i(\mathscr{T}_{R_{\mathscr{U}}}^\bullet(\widetilde{\mathfrak{D}}_{R_{\mathscr{U}}}(\mathbf{L}))$ is functorial for every $i \geq 0$ and we denote by $\mathscr{H}^i(\mathbf{L})$ the sheaf on the pointed étale site $\mathscr{X}_{\mathrm{et}}^\bullet$ associated to it. In §4 we prove Theorem 4.1: there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^q(\mathscr{X}_{\mathrm{et}}^\bullet, \mathscr{H}^p(\mathbf{L})) \Longrightarrow \mathrm{H}^{p+q}(X_{K,\mathrm{et}}, \mathbb{L}).$$

We view this result as a global analogue of $(*)$: the étale cohomology of $\mathbb{L}$ is calculated in terms of local relative (φ, Γ)-modules attached to $\mathbf{L}$.

The proof of Theorem 4.1 follows a roundabout path which was forced on us by lack of enough knowledge on étale cohomology of rigid analytic spaces. More precisely, for an algebraic, possibly infinite, extension M of K contained in $\overline{K}$, Faltings defines in [**14**] a Grothendieck topology $\mathfrak{X}_M$ on X (see also §4). The local system $\mathbb{L}$ may be thought of as a sheaf on $\mathfrak{X}_M$ and it follows from [**14**], see 4.4, that there is a natural isomorphism:

$$(**) \qquad \mathrm{H}^i(\mathfrak{X}_M, \mathbb{L}) \cong \mathrm{H}^i(X_{M,\mathrm{et}}, \mathbb{L}),$$

for all $i \geq 0$. The main tool for proving $(**)$ is the result: every point $x \in X_K$ has a neighborhood W which is $K(\pi, 1)$. Such a result, although believed to be true, is yet unproved in the rigid analytic setting. Therefore the proof of Theorem 4.1 goes as follows. Let $\mathbb{L}^{\mathrm{rig}}$ be the locally constant étale sheaf on X_K^{rig} associated to $\mathbb{L}$. We define the analogue Grothendieck topology $\widehat{\mathfrak{X}}_M$ on $\mathscr{X}$, prove that there is a spectral sequence with $E_2^{p,q} = \mathrm{H}^q(\mathscr{X}_{\mathrm{et}}^\bullet, \mathscr{H}^p(\mathbf{L}^{\mathrm{rig}}))$ abutting to $\mathrm{H}^{p+q}(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}})$, then compare $\mathrm{H}^i(\mathfrak{X}_M, \mathbb{L})$ to $\mathrm{H}^i(\widehat{\mathfrak{X}}_M, \mathbb{L}^{\mathrm{rig}})$ and in the end use Faltings' result $(**)$.

In §5 we introduce a certain family of continuous sheaves which we call *Fontaine sheaves* and which we denote by $\overline{\mathscr{O}}_{\_}$, $\mathscr{R}(\overline{\mathscr{O}}_{\_})$, $A_{\mathrm{inf}}^+(\overline{\mathscr{O}}_{\_})$. There are algebraic and analytic variants of these: the first are sheaves on $\mathfrak{X}_M$ and the second on $\widehat{\mathfrak{X}}_M$. We would like to remark that the local sections of the Fontaine sheaves are very complicated and they are **not** relative Fontaine rings. Continuous cohomology of continuous