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TEST CONFIGURATION AND GEODESIC RAYS

by

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Dedicated to Professor J. P. Bourguignon,
with affection, gratitude and admiration

Abstract. — This paper presents recent research findings on the connection between test configuration and geodesic ray in Kähler metric space. The purpose was to gain insight on the degeneration of Kähler metrics along geodesic rays. A result associating every smooth test configuration a $C^{1,1}$ geodesic ray is proved and exemplified with toric degenerations. Furthermore, we show that the ¥ invariant agrees with Futaki invariant, thus acts as a good substitute in general $C^{1,1}$ geodesic rays without a background test configuration. Based on the assumption of simple test configuration, we extend Donaldson's correspondence between solutions of Monge-Ampère equation and holomorphic discs. Results indicate that Chen and Tian's analysis on Monge-Ampère equation via holomoprhic discs could apply in simple test configuration.

***Résumé* (Configuration de test et rayons géodésiques).** — Cet article présente les dernières découvertes sur la connexion entre la configuration de test et les rayons géodésiques dans les espaces métriques kähleriens. Un résultat qui associe à chaque configuration de test lisse un $C^{1,1}$-rayon géodésique est démontré, et nous fournissons des exemples avec des dégénérations toriques. D'autre part, nous montrons que l'invariant ¥ s'accorde avec celui de Futaki, et forme ainsi un bon substitut dans le cas de $C^{1,1}$-rayons géodésiques généraux sans configuration de test. En nous basant sur l'hypothèse d'une configuration de test simple, nous étendons la correspondance de Donaldson entre les solution de l'équation de Monge-Ampère et les disques holomorphes. Les résultats indiquent que l'analyse de Chen et Tian sur l'équation de Monge-Ampère par le biais des disques holomorphes pourrait s'applique dans les configurations de test simples.

1. Introduction

The purpose of this paper is to explore the connection between geodesic rays in the space of Kähler metrics and test configurations in algebraic manifold [**15**]. This

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is a continuation of [**9**] in some aspects. In [**7**], the first named author and E. Calabi proved that the space of Kähler potentials is a non-positive curved space in the sense of Alexanderov. As a consequence, they proved that for any given geodesic ray and any given Kähler potential outside of the given ray, there always exists a geodesic ray in the sense of metric distance (L^2 in the Kähler potentials) which initiates from the given Kähler potential and parallel to the initial geodesic ray. The initial geodesic ray, plays the role of prescribing an asymptotic direction for the new geodesic ray out of any other Kähler potential. When the initial geodesic ray is smooth and is tamed by a bounded ambient geometry, the first named author [**9**] proved the existence of relative $C^{1,1}$ geodesic ray from any initial Kähler potential. (These definitions can be found in Section 2.) Similarly, as remarked in [**9**], a test configuration should play a similar role. One would like to know if it induces a relative $C^{1,1}$ geodesic ray from any other Kähler potential in the direction of test configuration. In [**3**], Arezzo and Tian proved a surprising result that for a smooth test configuration with analytic (smooth) central fiber, there always exists a general fiber sufficiently closed to the central fiber, such that there exists a smooth geodesic ray initiated from that fiber metric, and be asymptotically closed to the test configuration (or approximating to some analytic metric in the central fiber). A natural question, motivated by Arezzo-Tian's work, is if there exists a relative geodesic ray from arbitrary initial Kähler metric which also reflects the same geometry (i.e., degenerations) of the underlying test configuration. In section 3, we prove

Theorem 1.1. — *Every smooth test configuration induces a relative $C^{1,1}$ geodesic ray from any Kähler potential in the given class.* (1)

Test configurations can be viewed as algebraic rays, which are geodesics in a finite dimensional subspace(with new metric) of space of Kähler metrics. The geodesic rays induced by a test configuration are the rays parallel to the algebraic ray. They automatically have bounded ambient geometry introduced by the first named author [**9**].

Theorem 1.2. — *For simple test configuration* (2)*, if the induced geodesic ray is smooth regular* (3)*, then the generalized Futaki invariant agrees with the ¥ invariant* (4).

In 1982, E. Calabi asked if there always exists an extremal Kähler metric in every Kähler class [**5**]. This is a very ambitious conjecture which includes his famous conjecture on Kähler Einstein metric (when the first Chern class has a definite sign) as

(1) Following ideas of [**9**], the smooth assumption can be reduced to a lower bound of the Riemannian curvature of the total space.
(2) Definition 2.3.
(3) Definition 2.1, it is also equivalent to Definition 6.2 in this case.
(4) The ¥ invariant is defined by the first named author [**9**].

a special case. It was soon pointed out by Levine [**19**] that Calabi's conjecture can not hold for general Kähler class. However, it is understood among the experts that, with some modification, Calabi's conjecture might hold for general Kähler manifolds. Unfortunately, it is truely subtle and elusive to search/fromulate a correct statement regarding the existence of constant scalar curvature Kähler (cscK) metrics.

The generalized Futaki invariant or algebraic Futaki invariant is an algebraic notion which relates to the stability of projective manifolds. In the late 1990s, S. T. Yau conjectured that the existence of Kähler Einstein metrics in Fano manifolds is equivalent to some form of Stability of the underlying polarized Kähler class. Even though what stability notion to use is also part of puzzle, this is indeed a fundamental conjecture with respect to Kähler Einstein metrics. According to G. Tian [**34**] and Donaldson [**12**], this equivalence relation should be extended to include the case of the constant scalar curvature (cscK) metric in a general Kähler class. In [**34**] , G. Tian introduced the notion of K-Stability and in the same paper, he proved that the existence of KE metric implies weak K stability. In [**13**], Donaldson proved that, in algebraic manifold with discrete automorphism group, the existence of cscK metrics implies that the underlying Kähler class is Chow-Stable. In this paper, Donaldson actually formulated a new version (but equivalent) of K-Stability in terms of weights of Hilbert points. In Kähler toric varieties, the existence of cscK metrics implies that the underlying Kähler class is Semi-K stable [**15**]. Now it is a well-known conjecture that the existence of constant scalar curvature metrics, is equivalent to the K stability of the underlying complex polarization (the so called "Yau-Tian-Donaldson conjecture").

In [**9**], the first named author used the ¥ invariant to define geodesic stability. Theorem 1.2 states that geodesic stability in the algebraic manifold, is a proper generalization of K stability, at least conceptually. The first named author believes that the existence of KE metrics is equivalent to the geodesic stability introduced in [**9**]. Note that the geodesic stability introduced in [**9**] is a mild modification of a similar concept of S. K. Donaldson [**12**].

The Yau-Tian-Donaldson conjecture is a central problem in Kähler geometry now. Through the hard work of many mathematicians, we now know more about one direction (from existence to stability), cf. Tian [**34**], Donaldson [**16**] , Mabuchi [**22**], Paul-Tian [**23**], Phong-Sturm [**24**], Chen-Tian [**10**]... But on the direction from algebraic stability to existence, few progress has been made though. However, in toric manifolds, there has been special results of Donaldson [**15**] and Zhou-Zhu [**37**].

There is a recent intriguing work by V.Apostolov, D.Calderbank, P.Gauduchon and C.W.Tonnesen-Friedman [**2**]. They constructed an example which is suspected to be algebraically K stable [5], but admits no extremal Kähler metric. Perhaps one might speculate that, the geodesic stability aforementioned is one of the possible alternatives since it appears to be stronger than K stability and it is a non algebraic notion in nature.

The converse to Theorem 1.1 is widely open. In other words, it is hard to compactify a geodesic ray. The rays induced by any test configuration is very special in many aspects. For instance, generally speaking, the foliation of a smooth geodesic ray is a family of open strips which cover the base punctured disc. However, for the smooth geodesic rays induced from a test configurations, the strips always close up as punctured disc, or we may say that, the orbits are periodic. Unfortunately, having a periodic orbit does not appear to be enough to construct a test configuration. It would be a very intriguing problem to find a sufficient condition so that we can "construct" a test configuration out of a "good" geodesic ray.

Question A. — *Is there a canonical method to construct some test configuration/algebraic ray such that it reflects the same degeneration of a geodesic ray? What is natural geometric conditions on the "good" geodesic ray?*

Our second main result is to establish the correspondence between smooth regular solutions of Homogeneous complex Monge-Ampère equation (HCMA) on simple test configurations and some family of holomorphic discs in an ambient space $\mathcal{W}$ which will be explicitly constructed. We prove, in section 5:

Theorem 1.3. — *There is a one to one correspondence between smooth regular solutions of HCMA on simple test configuration $\mathcal{M}$ and families of holomorphic discs in $\mathcal{W}$ with proper boundary condition.* [6]

Note that in the case of disc, S. K. Donaldson [**14**] and S. Semmes [**30**] established first such a correspondence between the regularity of the solution of the HCMA equation and the smoothness of the moduli space of holomorphic discs whose boundary lies in some totally real sub-manifold. The theorem above is a generalization of Donaldson's result. Following this point of view, the regularity of the solution is essentially the same as the smoothness of the moduli space of these holomorphic discs under perturbation. As in [**14**], we proved the openness of smooth regular solutions in Section 6.

Theorem 1.4. — *Let $\rho(t)$ be a smooth regular geodesic ray induced by a simple test configuration. Then there exists a parallel smooth regular geodesic ray for any initial point sufficiently close to $\rho(0)$ in C^∞ sense.*

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(5) Generalized K stable for extremal Kähler metrics, cf. [**32**].

(6) In a followup work, we expect to extend this to all smooth test configurations.