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FLEXIBILITY OF SINGULAR EINSTEIN METRICS

by

Rafe Mazzeo

Dedicated to Jean Pierre Bourguignon on his 60th birthday.

Abstract. — This is a survey of a collection of related results about the deformation properties of Einstein metrics on a certain class of spaces with stratified singular structure. The results in low dimensions are particularly clean, and are motivated by applications in hyperbolic and convex geometry. The three-dimensional setting is related to an old conjecture by Stoker about flexibility of convex hyperbolic polyhedra, and we report on a partial answer. We also review some of the analytic methods used to prove these results.

Résumé **(Flexibilité des métriques d'Einstein singulières).** — Cet article constitue un compte-rendu d'une collection de résultats autour des propriétés de déformation des métriques d'Einstein sur une certaine classe d'espaces à structure singulière stratifiée. Les résultats en basse dimension sont particulièrement intéressants, et ils sont motivés par des applications en géométrie hyperbolique et convexe. La configuration 3-dimensionnelle est reliée à une vieille conjecture de Stoker sur la flexibilité des polyèdres convexes hyperboliques et nous proposons une réponse partielle. Nous examinons également certaines méthodes analytiques utilisées pour démontrer ces résultats.

1. Introduction

The construction and study of canonical metrics on smooth Riemannian manifolds is a longstanding central theme in geometric analysis. The term 'canonical' can be interpreted in many ways; we shall take it here to mean Einstein, so we study metrics satisfying $\mathrm{Ric}^g = \lambda g$ for some constant λ. Beyond the basic existence questions, one of the main problems in this subject is to understand whether a given Einstein metric is rigid or flexible, i.e. admits nontrivial deformations amongst Einstein metrics.

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As a rule of thumb, negative curvature usually implies rigidity of while positive (or even just nonnegative) curvature often allows 'flexibility'. Our goal here is to discuss how Einstein metrics on a certain class of stratified singular spaces are sometimes flexible precisely because of the geometry of the singular set. There are several well-known instances of this, for example the classical problem of determining the flexibility of convex polyhedra in space forms, to which some of the theory discussed below is directly applicable. This provides one motivation for the more general study of Einstein metrics on stratified spaces proposed here.

This paper is intended as a brief survey of some small part of a broader subject, focusing on one interesting class of stratified spaces – the iterated cone-edge spaces – and presenting some recent results about the local deformation theory of Einstein metrics on these, particularly in low dimensions where it is closely related to many investigations in geometric topology concerning the class of 'cone-manifolds' introduced by Thurston. The new results reported here are parts of various ongoing collaborations by the author with Gregoire Montcouquiol, Frank Pacard and Hartmut Weiss, and are also very closely related to his work with Olivier Biquard. The intention is to indicate the beginnings of a coherent 'story', and one which seems worthy of further development, albeit from a very personal point of view. Due to limitations of space and the author's expertise, we do not touch on many interesting situations where singular Einstein metrics have already been studied by others, e.g. for metrics with special holonomy, particularly in complex geometry. Finally, we also do not discuss any global aspects of this moduli problem, in particular the compactification theory, though this is likely to be both important and very interesting.

Let us first mention a few facts about Einstein metrics on smooth manifolds. Recall that a deformation of an Einstein metric g is a (smooth) one-parameter family of metrics g_t with $g_0 = g$; it is called a trivial deformation if there exists a one-parameter family of diffeomorphisms ϕ_t of the underlying manifold such that $g_t = \phi_t^* g_0$. In other words, the moduli space $\mathcal{E}(M)$ of Einstein metrics on a given manifold M is the space of all metrics satisfying the Einstein condition modulo diffeomorphisms. (Just as for surfaces, one may mod out by all diffeomorphisms or by those isotopic to the identity, but since our focus is on local aspects of the deformation problem, we do not emphasize this here.) As usual, it is more convenient to study an auxiliary equation whose solution space yields all (nearby) Einstein metrics without the diffeomorphism redundancy; this is done by introducing an auxiliary gauge condition to make the problem elliptic; we describe this later. There is a well-known result due to Koiso [**22**] which states that if M is compact, then $\mathcal{E}(M)$ is a finite dimensional analytic set. (This means that it can be covered by neighbourhoods, each of which is identified by a real analytic diffeomorphism with the zero set of an analytic function in a finite dimensional Euclidean space.) The subtlety in proving this, and the reason that its conclusion is not more specific (e.g. with respect to the dimension or smoothness of the moduli space) is that when the manifold is compact, the deformation theory is either trivial or obstructed. Indeed, one standard approach to such a result is to apply an implicit function theorem, for which one needs surjectivity of the linearization of the relevant operator, and if this holds, then the space of solutions of the nonlinear geometric problem is locally parametrized by elements of the nullspace of this linearization. (We describe this in greater detail in §6 below.) This linearization is a self-adjoint elliptic operator, so when M is compact, its surjectivity is equivalent to its injectivity. Thus if the linearization is surjective, it is injective too and the Einstein metric is rigid; on the other hand, if the linearization has nontrivial nullspace, then it has cokernel too, so the implicit function theorem does not directly apply. There is a standard trick to handle situations of this sort, known as Ljapunov-Schmidt reduction, but one can then only deduce much less precise conclusions.

Despite the fact that the 'formal dimension' of the moduli space of Einstein metrics on a compact manifold is zero, there are many manifolds M for which $\mathcal{E}(M)$ is positive dimensional and sometimes even smooth. The best known examples in higher dimensions are the families of flat tori, and less trivially, the family of Calabi-Yau K3 surfaces, for which very detailed results may be obtained using algebraic geometric techniques. Also worthy of note are the recent results of [**11**] about the existence of smooth high dimensional families of Einstein metrics on the sphere, all far from the standard metric, obtained using an integrable systems approach. On the other hand, as suggested above, if M is compact and the sectional curvatures of g are everywhere nonpositive, and negative somewhere, then g is rigid; this can be proved using the Bochner technique. The special case where (M, g) is locally symmetric was proved in various settings of increasing generality by Weil, Calabi, and Matsushita-Murakami. For quite different reasons it is known that the standard metric on the sphere is also rigid. We refer to the outstanding expository monographs [**6**], [**21**], and the collection [**23**], for more about these facts and their proofs.

When the manifold is noncompact or incomplete, this rigidity or deformation theory has a very different flavour. At one end is the study of the asymptotic boundary problems associated to Einstein metrics with certain types of asymptotically symmetric geometries, in particular the much-studied case of asymptotically hyperbolic Einstein metrics (also called Poincaré-Einstein metrics), see [**17**], [**24**], [**2**], as well as the work by Biquard on complex and quaternionic analogues, [**8**], [**7**], and more recently, some 'higher rank' analogues studied by the author in collaboration wtih Biquard, [**9**], [**10**]. As the name 'asymptotic boundary problem' suggests, complete Einstein metrics with these various types of asymptotic conditions come in infinite dimensional families, and the emphasis changes to parametrizing them by some appropriate type of boundary data, which in these cases are the associated parabolic geometries on the ideal boundary at infinity. The parabolic geometry associated to an asymptotically real hyperbolic Einstein metric is a conformal structure on the boundary of the geodesic compactification. The classical analogue of this, when $\dim M = 3$ and M is a quasiFuchsian convex cocompact hyperbolic manifold, was developed by Ahlfors and Bers; here, hyperbolic structures satisfying these hypotheses are in bijective correspondence with the space of conformal structures on the boundary at infinity, which is then a compact surface with two components. In the asymptotically complex or quaternionic settings, the parabolic geometries are the CR and quaternionic CR structures; for the higher rank cases, the relevant asymptotic boundary structures are somewhat less familiar but quite explicit, see [**9**]. There is also some recent progress by Anderson on the boundary problem in the usual sense for incomplete Einstein metrics on manifolds with boundary [**3**].

Of a different nature is the study of Einstein metrics which are asymptotically locally Euclidean (ALE), or which satisfy other more intricate but related asymptotic conditions, but which in any case are complete and have polynomial volume growth. Almost all known examples of these are metrics with restricted holonomy group, e.g. Kähler-Einstein or even hyperKähler, and that extra structure provides a substantial key to unlocking their properties. These have only finite dimensional deformation spaces, which are in some cases very well understood; we refer again to [**21**] for more on this.

On the other hand, there does not seem to have been any systematic study of Einstein metrics on various classes of spaces with 'geometrically structured' singularities, e.g. manifolds with conic points, edges and iterated edges, or more general stratified spaces, despite their ubiquity 'in nature'. As indicated above, we focus on the local rigidity/flexibity question, and in particular how geometric data at the singular locus can provide at least some of the moduli parameters. There is nothing approaching a comprehensive understanding of this phenomenon yet; rather, we simply present several recent results in this area in order to explain what is possible with current techniques and to emphasize this as an interesting area of study.

To be more specific, we first recall a particular class of Riemannian stratified spaces obtained by an iterated coning procedure and a class of Riemannian metrics on their principal smooth strata which induce metrics on each of the substrata. The general problem we pose is to study Einstein metrics in this class of singular spaces. In successive sections we consider this problem in the two, three, and higher-dimensional settings. Not surprisingly, the results are of decreasing specificity. The case of conic surfaces is certainly well-motivated through its association with marked Teichmüller theory, and serves as an excellent test-case for refining techniques for the more general settings. The results on this discussed here are joint work with H. Weiss. The