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ON UNIQUENESS OF STATIONARY VACUUM BLACK HOLES

by

Piotr T. Chruściel & João Lopes Costa

It is a pleasure to dedicate this work to J.-P. Bourguignon on the occasion of his 60th birthday.

Abstract. — We prove uniqueness of the Kerr black holes within the connected, non-degenerate, analytic class of regular vacuum black holes.

Résumé (*Sur l'unicité de trous noirs stationnaires dans le vide*). — On démontre l'unicité de trous noirs de Kerr dans la classe de trous noirs connexes, analytiques, réguliers, non-dégénérés, solutions des équations d'Einstein du vide.

1. Introduction

It is widely expected that the Kerr metrics provide the only stationary, asymptotically flat, sufficiently well-behaved, vacuum, four-dimensional black holes. Arguments to this effect have been given in the literature [**12, 84**] (see also [**51, 77, 91**]), with the hypotheses needed not always spelled out, and with some notable technical gaps. The aim of this work is to prove a precise version of one such uniqueness result for analytic space-times, with detailed filling of the gaps alluded to above.

The results presented here can be used to obtain a similar result for electro-vacuum black holes (compare [**13, 71**]), or for five-dimensional black holes with three commuting Killing vectors (see also [**56, 57**]); this will be discussed elsewhere [**31**].

We start with some terminology. The reader is referred to Section 2.1 for a precise definition of asymptotic flatness, to Section 2.2 for that of a domain of outer communications $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, and to Section 3 for the definition of mean-non-degenerate horizons. A Killing vector K is said to be complete if its orbits are complete, i.e., for every $p \in \mathscr{M}$ the orbit $\phi_t[K](p)$ of K is defined for all $t \in \mathbb{R}$; in an asymptotically flat context, K is called *stationary* if it is timelike at large distances.

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A key definition for our work is the following:

Definition 1.1. — *Let $(\mathscr{M}, \mathfrak{g})$ be a space-time containing an asymptotically flat end $\mathscr{S}_{\mathrm{ext}}$, and let K be stationary Killing vector field on $\mathscr{M}$. We will say that $(\mathscr{M}, \mathfrak{g}, K)$ is I^+–*regular *if K is complete, if the domain of outer communications $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$ is globally hyperbolic, and if $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$ contains a spacelike, connected, acausal hypersurface $\mathscr{S} \supset \mathscr{S}_{\mathrm{ext}}$, the closure $\overline{\mathscr{S}}$ of which is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotic ends, such that the boundary $\partial\overline{\mathscr{S}} := \overline{\mathscr{S}} \setminus \mathscr{S}$ is a topological manifold satisfying*

$$\partial\overline{\mathscr{S}} \subset \mathscr{E}^+ := \partial\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle \cap I^+(\mathscr{M}_{\mathrm{ext}}), \tag{1.1}$$

with $\partial\overline{\mathscr{S}}$ meeting every generator of $\mathscr{E}^+$ precisely once. (See Figure 1.1.)

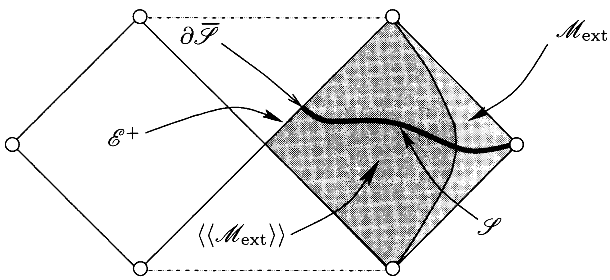

FIGURE 1.1. The hypersurface $\mathscr{S}$ from the definition of I^+–regularity.

In Definition 1.1, the hypothesis of asymptotic flatness is made for definiteness, and is not needed for several of the results presented below. Thus, this definition appears to be convenient in a wider context, e.g. if asymptotic flatness is replaced by Kaluza-Klein asymptotics, as in [**20, 23**].

Some comments about the definition are in order. First we require completeness of the orbits of the stationary Killing vector because we need an action of $\mathbb{R}$ on $\mathscr{M}$ by isometries. Next, we require global hyperbolicity of the domain of outer communications to guarantee its simple connectedness, to make sure that the area theorem holds, and to avoid causality violations as well as certain kinds of naked singularities in $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$. Further, the existence of a well-behaved spacelike hypersurface gives us reasonable control of the geometry of $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$, and is a prerequisite to any elliptic PDEs analysis, as is extensively needed for the problem at hand. The existence of compact cross-sections of the future event horizon prevents singularities on the future part of the boundary of the domain of outer communications, and eventually guarantees the smoothness of that boundary. (Obviously I^+ could have been replaced by I^-

throughout the definition, whence $\mathscr{E}^+$ would have become $\mathscr{E}^-$.) We find the requirement (1.1) somewhat unnatural, as there are perfectly well-behaved hypersurfaces in, e.g., the Schwarzschild space-time which do not satisfy this condition, but we have not been able to develop a coherent theory without assuming some version of (1.1). Its main point is to avoid certain zeros of the stationary Killing vector K at the boundary of $\mathscr{S}$, which otherwise create various difficulties; e.g., it is not clear how to guarantee then smoothness of $\mathscr{E}^+$, or the static-or-axisymmetric alternative. (1) Needless to say, all those conditions are satisfied by the Schwarzschild, Kerr, or Majumdar-Papapetrou solutions.

We have the following, long-standing conjecture, it being understood that both the Minkowski and the Schwarzschild space-times are members of the Kerr family:

Conjecture 1.2. — *Let $(\mathscr{M}, \mathfrak{g})$ be a vacuum, four-dimensional space-time containing a spacelike, connected, acausal hypersurface $\mathscr{S}$, such that $\overline{\mathscr{S}}$ is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotically flat ends. Suppose that there exists on $\mathscr{M}$ a complete stationary Killing vector K, that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and that $\partial\overline{\mathscr{S}} \subset \mathscr{M} \setminus \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Then $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Kerr space-time.*

In this work we establish the following special case thereof:

Theorem 1.3. — *Let $(\mathscr{M}, \mathfrak{g})$ be a stationary, asymptotically flat, I^+-regular, vacuum, four-dimensional analytic space-time. If each component of the event horizon is mean non-degenerate, then $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of one of the Weinstein solutions of Section 6.7. In particular, if $\mathscr{E}^+$ is connected and mean non-degenerate, then $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Kerr space-time.*

In addition to the references already cited, some key steps of the proof are due to Hawking [**48**], and to Sudarsky and Wald [**89**], with the construction of the candidate solutions with several non-degenerate horizons due to Weinstein [**93, 94**]. It should be emphasized that the hypotheses of analyticity and non-degeneracy are highly unsatisfactory, and one believes that they are not needed for the conclusion.

One also believes that no candidate solutions with more than one component of $\mathscr{E}^+$ are singularity-free, but no proof is available except for some special cases [**69, 92**].

A few words comparing our work with the existing literature are in order. First, the event horizon in a smooth or analytic black hole space-time is a priori only a Lipschitz surface, which is way insufficient to prove the usual static-or-axisymmetric alternative.

(1) In fact, this condition is not needed for *static* metric if, e.g., one assumes at the outset that all horizons are non-degenerate, as we do in Theorem 1.3 below, see the discussion in the Corrigendum to [**18**].

Here we use the results of [**22**] to show that event horizons in regular stationary black hole space-times are as differentiable as the differentiability of the metric allows. Next, no paper that we are aware of adequately shows that the "area function" is non-negative within the domain of outer communications; this is due both to a potential lack of regularity of the intersection of the rotation axis with the zero-level-set of the area function, and to the fact that the gradient of the area function could vanish on its zero level set *regardless of whether or not the event horizon itself is degenerate.* The second new result of this paper is Theorem 5.4, which proves this result. The difficulty here is to exclude *non-embedded Killing prehorizons* (for terminology, see below), and we have not been able to do it without assuming analyticity or axisymmetry, *even for static solutions.* Finally, no previous work known to us establishes the behavior, as needed for the proof of uniqueness, of the relevant harmonic map at points where the horizon meets the rotation axis. The third new result of this paper is Theorem 6.1, settling this question for non-degenerate black-holes. (This last result requires, in turn, the Structure Theorem 4.5 and the Ergoset Theorem 5.24, and relies heavily on the analysis in [**19**].) Last but not least, we provide a coherent set of conditions under which all pieces of the proof can be combined to obtain the uniqueness result.

We note that various intermediate results are established under conditions weaker than previously cited, or are generalized to higher dimensions; this is of potential interest for further work on the subject.

1.1. Static case. — Assuming *staticity*, i.e., stationarity and hypersurface-orthogonality of the stationary Killing vector, a more satisfactory result is available in space dimensions less than or equal to seven, and in higher dimensions on manifolds on which the Riemannian rigid positive energy theorem holds: non-connected configurations are excluded, without any *a priori* restrictions on the gradient $\nabla(\mathfrak{g}(K,K))$ at event horizons.

More precisely, we shall say that a manifold $\widehat{\mathscr{S}}$ is of *positive energy type* if there are no asymptotically flat complete Riemannian metrics on $\widehat{\mathscr{S}}$ with positive scalar curvature and vanishing mass except perhaps for a flat one. This property has been proved so far for all n–dimensional manifolds $\widehat{\mathscr{S}}$ obtained by removing a finite number of points from a compact manifold of dimension $3 \leq n \leq 7$ [**86**], or under the hypothesis that $\widehat{\mathscr{S}}$ is a spin manifold of any dimension $n \geq 3$, and is expected to be true in general [**14, 70**].

We have the following result, which finds its roots in the work of Israel [**61**], with further simplifications by Robinson [**85**], and with a significant strengthening by Bunting and Masood-ul-Alam [**10**]: