Astérisque

Hideki Omori
Yoshiaki Maeda
Naoya Miyazaki
Akira Yoshioka

A new nonformal noncommutative calculus: associativity and finite part regularization

Astérisque, tome 321 (2008), p. 267-297
<http://www.numdam.org/item?id=AST_2008__321__267_0>

Astérisque
321, 2008, p. 267–297

A NEW NONFORMAL NONCOMMUTATIVE CALCULUS: ASSOCIATIVITY AND FINITE PART REGULARIZATION

by

Hideki Omori, Yoshiaki Maeda, Naoya Miyazaki & Akira Yoshioka

Abstract. — We interpret the element $\frac{1}{2i\hbar}(u * v + v * u)$ in the generators u, v of the Weyl algebra W_2 as an indeterminate in $\mathbb{N} + \frac{1}{2}$ or $-(\mathbb{N} + \frac{1}{2})$, using methods of the transcendental calculus outlined in the announcement [**13**]. The main purpose of this paper is to give a rigorous proof for the part of [**13**] which introduces this indeterminate phenomenon. Namely, we discuss how to obtain associativity in the transcendental calculus and show how the Hadamard finite part procedure can be implemented in our context.

***Résumé* (Un nouveau calcul non-formel et non-commutatif : associativité et régularisation des parties finies)**

Nous interprêtons l'élément $\frac{1}{2i\hbar}(u*v+v*u)$ dans les générateurs u, v de l'algèbre de Weyl W_2 en tant qu'indéterminés dans $\mathbb{N} + \frac{1}{2}$ ou $-(\mathbb{N} + \frac{1}{2})$, en utilisant des méthodes du calcul transcendental décrit dans l'annonce [**13**]. Le but principal de cet article est de donner une preuve rigoureuse de la partie de [**13**] qui introduit ce phénomène indéterminé. À savoir, nous discutons la manière d'obtenir l'associativité dans le calcul transcendental et de montrer comment la procédure de parties finies de Hadamard peut être implémentée dans notre contexte.

1. Introduction

Deformation quantization, first proposed in [**3**], is a fruitful approach to developing quantum theory in a purely algebraic framework, and was a prototype for noncommutative calculus on noncommutative spaces. It was first treated as a formal noncommutative calculus, with the Planck constant $\hbar$ regarded as a formal parameter, but has been extended to nonformal cases, as in the studies of noncommutative tori

2000 Mathematics Subject Classification. — 53D55, 53D10; 46L65.
Key words and phrases. — Weyl algebra, transcendental calculus.

Partially supported by Grant-in-Aid for Scientific Research #18204006 (Y. Maeda), #18540093 (N. Miyazaki), #19540103 (A. Yoshioka) by the Ministry of Education, Science and Culture, Japan.

[**18**] and quantum groups [**20**]. In fact, the formal and nonformal noncommutative calculus have quite different features.

In [**12**], we analyzed star exponential functions of quadratic forms in the Weyl algebra and uncovered several mysterious phenomena unanticipated from the formal case. These mysterious phenomena reflect the fact that star exponential functions of quadratic forms (see [**11**] and [**15**]) lie outside of the Weyl algebra. These new features suggest a new approach to noncommutative nonformal calculus. In this paper, we show that this new calculus is necessary to treat transcendental elements of the Weyl algebra.

From the papers [**12**]-[**13**], we know that the Moyal product, the most typical product on the Weyl algebra, is not sufficient to treat transcendental elements such as star exponential functions. For this reason, we introduced a family of $*_K$-products on the Weyl algebra depending on a complex symmetric matrix K and developed a transcendental nonformal noncommutative calculus specifically formulated to treat star exponential functions of quadratic forms. The transcendental elements of the Weyl algebra have a realization depending on the $*_K$-product, which we called the *K-ordered expression.* Thus, properties of (transcendental) elements of the Weyl algebra depend on the choice of product $*_K$,

We now propose as a principle, called the *Independence of Ordering Principle* (IOP), that the relevant properties of transcendental elements of the Weyl algebra do not depend on the choice of ordered expression, just as properties and objects in differential geometry do not depend on the choice of coordinate expression. Following this principle, in [**12**] we proposed the notion of a group-like object of star exponential functions of quadratic forms on the Weyl algebra. The IOP seems to be a new outlook on interpreting physical phenomena/mathematical phenomena, especially for treating quantum objects and phenomena from an algebraic point of view.

As a test case, we examine this principle on the nonformal noncommutative calculus for transcendental elements of the Weyl algebra. As part of this approach, we interpret an element as an indeterminate in a discrete set in the case of the Weyl algebra with two generators.

Let W_2 be the Weyl algebra with generators u, v obeying the commutation relation

$$(1) \qquad [u,v] = -i\hbar.$$

We consider the element $\frac{1}{i\hbar}u \circ v= \frac{1}{2i\hbar}(u * v + v * u)$ of W_2. We show that $\frac{1}{i\hbar}u \circ v$ can be interpreted as an indeterminate in $\mathbb{N} + \frac{1}{2}$ or $-(\mathbb{N} + \frac{1}{2})$, not from a more standard operator theoretic point of view but from a purely algebraic approach, using the IOP that a physical/mathematical object should be independent of its various ordered expressions.

In our approach, we interpret $\frac{1}{i\hbar}u\circ v$ in two ways: 1) via the analytic continuation of inverses of $z+\frac{1}{i\hbar}u\circ v$ and 2) via the $*$-product of the $*$-sin function and the $*$-gamma function using ordered expressions. These results have been already announced in [**13**] with outlines of proofs. The main purpose of this paper is to give a rigorous description of method 1) and therefore to realize $\frac{1}{i\hbar}u\circ v$ as an indeterminate in the discrete set. The main ingredients of the proof are dealing with associativity in the framework of the transcendental calculus of [**13**] and applying the Hadamard finite part procedure in this context. For a family of $*_K$-products on the Weyl algebra W_2, we provide rules for the associativity of the extended products $*_K$, and in preparation for the definition of the inverse of $z+\frac{1}{i\hbar}u\circ v$, we investigate star exponentials $e_*^{z+\frac{1}{i\hbar}u\circ v}$ and their ordered versions.

We leave the the finite part regularization method for Fréchet algebra valued functions in the subsection 6.1. For a holomorphic function $f(z)$ with a pole at $z=z_0$, we define the finite part of $f(z)$ as

$$\mathrm{FP}(f(z))=\begin{cases} f(z) & z\neq z_0\\ \mathrm{Res}_{w=0}\frac{1}{w}(f(z_0+w)) & z=z_0.\end{cases}$$

We first construct the inverses of $z+\frac{1}{i\hbar}u\circ v$ by using the star exponential function $e_*^{z+\frac{1}{i\hbar}u\circ v}$ and a K-ordered expression. We can construct two inverses of $z+\frac{1}{i\hbar}u\circ v$ as follows:

$$(z+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}=\int_{-\infty}^{0}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$$

and

$$(z+\frac{1}{i\hbar}u\circ v)_{*-}^{-1}=-\int_{0}^{\infty}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$$

(see [**7**] and [**10**] for more details). Both inverses have analytic continuations for generic ordered expression. In §6, we mainly study the inverse $(z+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}$, as the other inverse has similar properties.

In §6, we show the following:

Theorem 1.1. — *For generic ordered expressions, the inverses* $(z+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}$, $(z-\frac{1}{i\hbar}u\circ v)_{*-}^{-1}$ *extend to* $\mathcal{E}_{2+}(\mathbb{C}^2)$*-valued holomorphic functions of* z *on* $\mathbb{C}-\{-(\mathbb{N}+\frac{1}{2})\}$.

Here, we refer the class $\mathcal{E}_{2+}(\mathbb{C}^2)$ in the subsection 2.2.

Employing the Hadamard technique of extracting finite parts of divergent integrals, we now extend the definition of the $*$-product using finite part regularization. We define the new product of $(z+\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1}$ with either the polynomial $q(u,v)$ or $q(u,v)=e_*^{s\frac{1}{i\hbar}u\circ v}$ by

$$(z+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1}\tilde{*}q(u,v)=(\mathrm{FP}(z+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1})*q(u,v). \tag{2}$$

Note that the result may not be continuous in z.

The following is an description of the discrete phenomena for $\frac{1}{i\hbar}u\circ v$ via method 1):

Theorem 1.2. — *Using definition* (2) *for the* $\tilde{*}$*-product, we have*

$$(3)\qquad (z+\tfrac{1}{i\hbar}u\circ v)\tilde{*}(z+\tfrac{1}{i\hbar}u\circ v)^{-1}_{*+} = \begin{cases} 1 & z\notin -(\mathbb{N}+\frac{1}{2}) \\ 1-\frac{1}{n!}(\frac{1}{i\hbar}u)^n * \varpi_{00} * v^n & z=-(n+\frac{1}{2}) \end{cases},$$

$$(4)\qquad (z-\tfrac{1}{i\hbar}u\circ v)\tilde{*}(z-\tfrac{1}{i\hbar}u\circ v)^{-1}_{*-} = \begin{cases} 1 & z\notin -(\mathbb{N}+\frac{1}{2}) \\ 1-\frac{1}{n!}(\frac{1}{i\hbar}v)^n * \overline{\varpi}_{00} * u^n & z=-(n+\frac{1}{2}) \end{cases}.$$

for generic ordered expressions.

We will interpret this discrete phenomena for $\frac{1}{i\hbar}u\circ v$ via method 2) in a forthcoming paper.

We would like to thank Steven Rosenberg and Sylvie Paycha for their suggestions about regularization methods.

Finally, we are honored to contribute our paper to the Proceedings for the 60th birthday celebration of Jean Pierre Bourguignon, whose friendship with us for over 20 years we warmly acknowledge.

2. General ordered expressions and IOP

We introduce a method to realize the Weyl algebra via a family of ordered expressions. This leads to a transcendental calculus for the Weyl algebra.

2.1. Fundamental product formulas and intertwiners.

— Let $\mathfrak{S}(n)$ and $\mathfrak{A}(n)$ be the spaces of complex symmetric matrices and skew-symmetric matrices respectively, and set $\mathfrak{M}(n)=\mathfrak{S}(n)\oplus\mathfrak{A}(n)$. We denote by $\boldsymbol{u}$ the set of generators $\boldsymbol{u}=(u_1,\dots,u_{2m})$. For an arbitrary fixed $n\times n$-complex matrix $\Lambda\in\mathfrak{M}(n)$, we define a product $*_{_\Lambda}$ on the space of polynomials $\mathbb{C}[\boldsymbol{u}]$ by the formula

$$(5)\quad f *_{_\Lambda} g = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}}\Lambda^{ij}\overrightarrow{\partial_{u_j}})} g = \sum_k \frac{(i\hbar)^k}{k!2^k}\Lambda^{i_1j_1}\cdots\Lambda^{i_kj_k}\partial_{u_{i_1}}\cdots\partial_{u_{i_k}} f\ \partial_{u_{j_1}}\cdots\partial_{u_{j_k}} g.$$

It is known and not hard to prove that $(\mathbb{C}[\boldsymbol{u}],*_{_\Lambda})$ is an associative algebra.

The algebraic structure of $(\mathbb{C}[\boldsymbol{u}],*_{_\Lambda})$ is determined by the skew-symmetric part of Λ, if the generators are fixed. In particular, if Λ is a symmetric matrix, $(\mathbb{C}[\boldsymbol{u}],*_{_\Lambda})$ is isomorphic to the usual polynomial algebra.