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KEFENG LIU
XIAOFENG SUN
SHING-TUNG YAU

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GEOMETRY OF MODULI SPACES

by

Kefeng Liu, Xiaofeng Sun & Shing-Tung Yau

Dedicated to Jean Pierre Bourguignon

Abstract. — In this paper we describe some recent results on the geometry of the moduli space of Riemann surfaces. We surveyed new and classical metrics on the moduli spaces of hyperbolic Riemann surfaces and their geometric properties. We then discussed the Mumford goodness and generalized goodness of various metrics on the moduli spaces and their deformation invariance. By combining with the dual Nakano negativity of the Weil-Petersson metric we derive various consequences such that the infinitesimal rigidity, the Gauss-Bonnet theorem and the log Chern number computations.

***Résumé* (Géométrie des espaces de modules).** — Dans cet article nous décrivons certains résultats récents en géométrie de l'espace de modules des surfaces de Riemann. Nous parcourons un certain nombre de métriques classiques et nouvelles sur les les espaces de modules de surfaces de Riemann hyperboliques et leur propriétés géométriques. Ensuite nous discutons la bonté de Mumford et la bonté généralisée de différentes métriques sur l'espace de modules et leurs invariance de déformation. En combinant avec la négativité de Nakano duale de la métrique de Weil-Peterson nous en tirons différentes conséquences telles que la rigidité infinitésimale, le théorème de Gauss-Bonnet et les calculs de nombres logarithmiques de Chern.

1. Introduction

In this paper we describe our recent work on the geometry of the moduli space of Riemann surfaces $\mathcal{M}_g$. We will survey the properties of the canonical metrics especially the asymptotic behavior.

This paper is organized as follows. In the second section we will briefly recall the deformation theory of Riemann surfaces. In the third section we will recall the

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Ricci and perturbed Ricci metrics as welll as the Kähler-Einstein metric which were discussed in [**5**] and [**6**].

In the fourth section we will discuss the notion of Mumford goodness and our generalizations to the p-goodness and intrinsic goodness. We then discuss the relation of the goodness and the complex Monge-Amperé equation as well as the Kähler-Ricci flow. In the last section we will discuss the applications of these fine properties of the canonical metrics.

2. Fundamentals of Teichmüller and Moduli Spaces

We briefly recall the fundamental theory of the geometry of Teichmüller and moduli spaces of hyperbolic Riemann surfaces in this section. Most of the results can be found in [**5**], [**6**], [**7**] and [**18**].

Let $\mathcal{M}_{g,k}$ be the moduli space of Riemann surfaces of genus g with k punctures such that $2g-2+k>0$. By the uniformization theorem we know there is a unique hyperbolic metric on such a Riemann surface. To simplify the computation, through out this paper, we will assume $k=0$ and $g\geq 2$ and work on $\mathcal{M}_g$. Most of the results can be trivially generalized to $\mathcal{M}_{g,k}$.

We first recall the local geometry of $\mathcal{M}_g$. For each point $s\in\mathcal{M}_g$, let X_s be the corresponding Riemann surface. By the Kodaira-Spencer deformation theory and Hodge theory, we know

$$T_s\mathcal{M}_g\cong H^1(X_s,T_{X_s})\cong H^{0,1}(X_s,T_{X_s}).$$

It follows direct from Serre duality that

$$T_s^*\mathcal{M}_g\cong H^0(X_s,K_{X_s}^2).$$

By the Riemann-Roch theorem, we know that the complex dimension of the moduli space is $n=\dim_{\mathbb{C}}\mathcal{M}_g=3g-3$. Given a Riemann surface X of genus $g\geq 2$, we denote by λ the unique hyperbolic (Kähler-Einstein) metric on X. Let z be local holomorphic coordinate on X. We normalize λ:

$$\partial_z\partial_{\bar z}\log\lambda=\lambda. \tag{2.1}$$

Let $\mathcal{T}_g$ be the Teichmüller space. It is well known that $\mathcal{T}_g$ is a domain of holomorphy and $\mathcal{M}_g$ is a quasi-projective orbifold. There are many canonical metrics on $\mathcal{T}_g$. These are the metrics where biholomorphisms are automatically isometries and thus these metrics descent down to $\mathcal{M}_g$.

There are three complex Finsler metrics on $\mathcal{T}_g$: The Teichmüller metric $\|\cdot\|_T$, the Kobayashi metric $\|\cdot\|_K$ and the Carathеódory metric $\|\cdot\|_C$. Each of these metrics defines a norm on the tangent space of $\mathcal{T}_g$. These metrics are non-Kähler. By the famous work of Royden we know that the Teichmüller metric coincides with the Kobayashi metric:

$$\|\cdot\|_T = \|\cdot\|_K.$$

We now describe the Kähler metrics. The first known Kähler metric is the Weil-Petersson metric ω_{WP}. Since $\mathcal{T}_g$ is a domain of holomorphy, there is a complete Kähler-Einstein metric on $\mathcal{T}_g$ due to the work of Cheng and Yau [**2**]. Since $\mathcal{M}_g$ is quasi-projective, there exist a Kähler metric on $\mathcal{M}_g$ with Poincaré growth. Furthermore, one has the Bergman metric associate to $\mathcal{T}_g$ and the Kähler metric defined by McMullen [**10**] by perturbing the Weil-Petersson metric.

In [**5**] and [**6**] we defined two new Kähler metrics: the Ricci and perturbed Ricci metrics which have very nice curvature and asymptotic properties. These metrics will be discussed in the following sections.

We now recall the construction of the Weil-Petersson metric. Let $(s_1, \cdots, s_n)$ be local holomorphic coordinates on $\mathcal{M}_g$ near a point p and let X_s be the corresponding Riemann surfaces. Let $\rho : T_s\mathcal{M}_g \to H^1(X_s, TX_s) \cong H^{0,1}(X_s, TX_s)$ be the Kodaira-Spencer map. Then the harmonic representative of $\rho\left(\frac{\partial}{\partial s_i}\right)$ is given by

$$\rho\left(\frac{\partial}{\partial s_i}\right) = \partial_{\bar{z}}\left(-\lambda^{-1}\partial_{s_i}\partial_{\bar{z}}\log\lambda\right)\frac{\partial}{\partial z}\otimes d\bar{z} = B_i. \tag{2.2}$$

If we let $a_i = -\lambda^{-1}\partial_{s_i}\partial_{\bar{z}}\log\lambda$ and let $A_i = \partial_{\bar{z}}a_i$, then the harmonic lift v_i of $\frac{\partial}{\partial s_i}$ is given by

$$v_i = \frac{\partial}{\partial s_i} + a_i\frac{\partial}{\partial z}. \tag{2.3}$$

The well-known Weil-Petersson metric $\omega_{WP} = \frac{\sqrt{-1}}{2}h_{i\bar{j}}ds_i \wedge d\bar{s}_j$ on $\mathcal{M}_g$ is the L^2 metric on $\mathcal{M}_g$:

$$h_{i\bar{j}}(s) = \int_{X_s} A_i\overline{A}_j\ dv \tag{2.4}$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\bar{z}$ is the volume form on X_s. It was proved by Ahlfors that the Ricci curvature of the Weil-Petersson metric is negative. The upper bound of the Ricci curvature of the Weil-Petersson metric was conjectured by Royden and was proved by Wolpert [**16**].

In our work [**5**] we defined the Ricci metric ω_τ:

$$\omega_\tau = -Ric\left(\omega_{WP}\right) \tag{2.5}$$

and the perturbed Ricci metric $\omega_{\widetilde{\tau}}$:

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP} \tag{2.6}$$

where C is a positive constant. These new Kähler metrics have good curvature and asymptotic properties and play important roles in out study.

Now we describe the curvature formulas of the Weil-Petersson metric. Please see [5] and [6] for details. We denote by $f_{i\bar{j}} = A_i\overline{A_j}$ where each A_i is the harmonic Beltrami differential corresponding to the local holomorphic vector field $\frac{\partial}{\partial s_i}$. It is clear that $f_{i\bar{j}}$ is a function on X. We let $\Box = -\partial_z\partial_{\bar{z}}$ be the Laplace operator, let $T = (\Box + 1)^{-1}$ be the Green operator and let $e_{i\bar{j}} = T(f_{i\bar{j}})$. The functions $e_{i\bar{j}}$ and $f_{i\bar{j}}$ are building blocks of these curvature formula.

Theorem 2.1. — *The curvature formula of the Weil-Petersson metric was given by*

$$R_{i\bar{j}k\bar{l}} = -\int_{X_s} (e_{i\bar{j}}f_{k\bar{l}} + e_{i\bar{l}}f_{k\bar{j}})\, dv. \tag{2.7}$$

This formula was first established by Wolpert [16] and was generalized by Siu [14] and Schumacher [13] to higher dimensions. A short proof can be found in [5].

It is easy to derive information of the sign of the curvature of the Weil-Petersson metric from its curvature formula (2.7). However, the Weil-Petersson metric is incomplete and its curvature has no lower bound. Thus we need to look at its asymptotic behavior. We now recall geometric construction of the Deligne-Mumford (DM) moduli space and the degeneration of hyperbolic metrics. Please see [5] and [16] for details.

Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$ and let $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. It was shown in [3] that D is a divisor with only normal crossings. A point $y \in D$ corresponds to a stable nodal surface X_y. A point $p \in X_y$ is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v) \mid uv = 0,\ |u|, |v| < 1\} \subset \mathbb{C}^2$. Let $p_1, \cdots, p_m \in X_y$ be the nodes. X_y is stable if each connected component of $X_y \setminus \{p_1, \cdots, p_m\}$ has negative Euler characteristic.

Fix a point $y \in D$, we assume the corresponding Riemann surface X_y has m nodes. Now for any point $s \in \mathcal{M}_g$ lying in a neighborhood of y, the corresponding Riemann surface X_s can be decomposed into the thin part which is a disjoint union of m collars and the thick part where the injectivity radius with respect to the Kähler-Einstein metric is uniformly bounded from below.

There are two kinds of local holomorphic coordinate on a collar or near a node. We first recall the rs-coordinate defined by Wolpert in [18]. In the node case, given a nodal surface X with a node $p \in X$, we let a, b be two punctures which are glued together to form p.

Definition 2.1. — *A local coordinate chart (U, u) near a is called rs-coordinate if $u(a) = 0$ where u maps U to the punctured disc $0 < |u| < c$ with $c > 0$, and the restriction to U of the Kähler-Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V, v) near b is defined in a similar way.*

In the collar case, given a closed surface X, we assume there is a closed geodesic $\gamma \subset X$ such that its length $l = l(\gamma) < c_*$ where c_* is the collar constant.