Astérisque

ROBERT L. BRYANT

Gradient Kähler Ricci solitons

Astérisque, tome 321 (2008), p. 51-97

<http://www.numdam.org/item?id=AST_2008__321__51_0>

© Société mathématique de France, 2008, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Astérisque
321, 2008, p. 51–97

GRADIENT KÄHLER RICCI SOLITONS

by

Robert L. Bryant

To Jean Pierre Bourguignon, on the occasion of his 60th *birthday.*

Abstract. — Some observations about the local and global generality of gradient Kähler Ricci solitons are made, including the existence of a canonically associated holomorphic volume form and vector field, the local generality of solutions with a prescribed holomorphic volume form and vector field, and the existence of Poincaré coordinates in the case that the Ricci curvature is positive and the vector field has a fixed point.

***Résumé* (Solitons gradients de Kähler-Ricci).** — Nous proposons quelques observations sur les généralités locale et globale des solitons gradients de Kähler-Ricci, y compris l'existence d'une forme de volume holomorphe et d'un champ de vecteurs canoniquement associés, la généralité locale de solutions pour une forme de volume holomorphe et un champ de vecteurs donnés, et l'existence de coordonnées de Poincaré dans le cas où la courbure de Ricci est positive et le champ de vecteurs a un point fixe.

1. Introduction and Summary

This article concerns the local and global geometry of gradient Kähler Ricci solitons, i.e., Kähler metrics g on a complex n-manifold M that admit a *Ricci potential*, i.e., a function f such that $\mathrm{Ric}(g) = \nabla^2 f$ (where ∇ denotes the Levi-Civita connection of M.

These metrics arise as limiting metrics in the study of the Ricci flow $g_t = -2\,\mathrm{Ric}(g)$ applied to Kähler metrics. Under the Ricci flow, a gradient Kähler Ricci soliton g_0

2000 Mathematics Subject Classification. — 53C55, 58G11.
Key words and phrases. — Ricci flow, solitons, normal forms.

Thanks to Duke University for its support via a research grant, to the NSF for its support via grants DMS-8905207, DMS-0103884, and DMS-0604195 and to Columbia University for its support via an Eilenberg Visiting Professorship.

© Astérisque 321, SMF 2008

evolves by flowing under the vector field ∇f, i.e.,

$$g(t) = {\exp_{(-t\nabla f)}}^*(g_0). \tag{1.1}$$

In particular, if the flow of ∇f is complete, then the Ricci flow with initial value g_0 exists for all time.

The reader who wants more background on these metrics might consult the references and survey articles [**3, 5, 10**]. The references [**8, 9, 6, 14**] contain further important work in the area and will be cited further below.

1.1. Basic facts. — Unless the metric g admits flat factors, the equation $\mathrm{Ric}(g) = \nabla^2 f$ determines f up to an additive constant and it does no harm to fix a choice of f for the discussion. For simplicity, it does no harm to assume that g has no (local) flat factors and so this will frequently be done. Also, the Ricci-flat case (aka the Calabi-Yau case), in which $\mathrm{Ric}(g) = 0$, is a special case that is usually treated by different methods, so it will usually be assumed that $\mathrm{Ric}(g) \neq 0$. (Indeed, most of the latter part of this article will focus on the case in which $\mathrm{Ric}(g) > 0$).

1.1.1. *The associated holomorphic vector field Z.* — One of the earliest observations [**2**] made about gradient Kähler Ricci solitons is that the vector field ∇f is the real part of a holomorphic vector field and that, moreover, $J(\nabla f)$ is a Killing field for g. In this article, I will take $Z = \frac{1}{2}\big(\nabla f - \mathrm{i}J(\nabla f)\big)$ to be the holomorphic vector field *associated* to g.

1.1.2. *The holomorphic volume form Υ.* — In the Ricci-flat case, at least when M is simply connected, it is well-known that there is a g-parallel holomorphic volume form Υ, i.e., one which satisfies the condition that $\mathrm{i}^{n^2} 2^{-n}\, \Upsilon \wedge \overline{\Upsilon}$ is the real volume form determined by g and the J-orientation.

In § 2.2, I note that, for any gradient Kähler Ricci soliton g with Ricci potential f defined on a simply connected M, there is a holomorphic volume form Υ (unique up to a constant multiple of modulus 1) such that $\mathrm{i}^{n^2} 2^{-n}\, \mathrm{e}^{-f}\, \Upsilon \wedge \overline{\Upsilon}$ is the real volume form determined by g and the J-orientation. Of course, Υ is not g-parallel (unless g is Ricci-flat) but satisfies $\nabla \Upsilon = \frac{1}{2}\, \partial f \otimes \Upsilon$.

This leads to a notion of *special* coordinate charts for (g, f) i.e., coordinate charts (U, z) such that the associated coordinate volume form $\mathrm{d}z = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$ is the restriction of Υ to U. In such coordinate charts, several of the usual formulae simplify for gradient Kähler Ricci solitons.

1.1.3. *The Υ-divergence of Z.* — Given a vector field and and volume form, the divergence of the vector field with respect to the volume form is well defined. It turns out to be useful to consider this quantity for Z and Υ. The divergence in this case is the (necessarily holomorphic) function h that satisfies $\mathsf{L}_Z\, \Upsilon = h\, \Upsilon$.

By general principles, the scalar function h must be expressible in terms of the first and second derivatives of f. Explicit computation (Proposition 4) yields

$$2h = \operatorname{tr}_g\big(\nabla^2 f\big) + |\nabla f|^2 = R(g) + |\nabla f|^2, \tag{1.2}$$

where $R(g) = \operatorname{tr}_g\big(\operatorname{Ric}(g)\big)$ is the scalar curvature of g. In particular, h is real-valued and therefore constant. Now, the constancy of $R(g) + |\nabla f|^2$ had already been noted and utilized by Hamilton and Cao [**6**]. However, its interpretation as a holomorphic divergence seems to be new.

1.2. Generality. — An interesting question is: How many gradient Kähler Ricci solitons are there? Of course, this rather vague question can be sharpened in several ways.

The point of view adopted in this article is to start with a complex n-manifold M already endowed with a holomorphic volume form Υ and a holomorphic vector field Z and ask how many gradient Kähler solitons on M there might be (locally or globally) that have Z and Υ as their associated holomorphic data.

An obvious necessary condition is that the divergence h of Z with respect to Υ must be a real constant.

1.2.1. *Nonsingular extension.* — Away from the singularities (i.e., zeroes) of Z, this divergence condition turns out to be locally sufficient.

More precisely, I show (see Theorem 2) that if $H \subset M$ is an embedded complex hypersurface that is transverse at each of its points to Z, and g_0 and f_0 are, respectively, a real-analytic Kähler metric and function on H, then there is an open neighborhood U of H in M on which there exists a gradient Kähler Ricci soliton g with potential f whose associated holomorphic quantities are Z and Υ and such that g and f pull back to H to become g_0 and f_0. The pair (g, f) is essentially uniquely specified by these conditions. The real-analyticity of the 'initial data' g_0 and f_0 is necessary in order for an extention to exist since any gradient Kähler Ricci soliton is real-analytic anyway (see Remark 4).

Roughly speaking, this result shows that, away from singular points of Z, the local solitons g with associated holomorphic data (Z, Υ) depend on two arbitrary (real-analytic) functions of $2n-2$ variables.

1.2.2. *Singular existence.* — The existence of (local) gradient Kähler solitons in a neighborhood of a singularity p of Z is both more subtle and more interesting.

Even if the divergence of Z with respect to Υ is a real constant, it is not true in general that a gradient Kähler Ricci solition with Z and Υ as associated holomorphic data exists in a neighborhood of such a p.

I show (Proposition 6) that a necessary condition is that there exist p-centered holomorphic coordinates $z = (z^i)$ on a p-neighborhood $U \subset M$ and real numbers $h_1, \ldots, h_n$ such that, on U,

$$Z = h_1\, z^1\, \frac{\partial}{\partial z^1} + \cdots + h_n\, z^n\, \frac{\partial}{\partial z^n}. \tag{1.3}$$

In other words, Z must be holomorphically linearizable, with real eigenvalues. (1)

In such a case, if $\mathsf{L}_Z\, \Upsilon = h\Upsilon$ where h is a constant, then $h = h_1 + \cdots + h_n$. I show (Proposition 7) that, moreover, in this case, one can always choose Z-linearizing coordinates as above so that $\Upsilon = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$.

Thus, the possible local singular pairs (Z, Υ) that can be associated to a gradient Kähler Ricci soliton are, up to biholomorphism, parametrized by n real constants.

Using this normal form, one then observes that, by taking products of solitons of dimension 1, any set of real constants $(h_1, \ldots, h_n)$ can occur (see Remark 9). Since, for any gradient Kähler Ricci soliton g with associated holomorphic data (Z, Υ), the formula $\mathrm{Ric}(g) = \mathsf{L}_{\mathrm{Re}(Z)}\, g$ holds, it follows that if g is such a Kähler Ricci soliton defined on a neighborhood of a point p with $Z(p) = 0$, then $h_1, \ldots, h_n$ are the eigenvalues (each of even multiplicity) of $\mathrm{Ric}(g)$ with respect to g at p.

However, this does not fully answer the question of how 'general' the solitons are in a neighborhood of such a p. In fact, this very subtly depends on the numbers h_i. For example, if the $h_i \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then any gradient Kähler Ricci soliton g with associated data (Z, Υ) defined on a neighborhood of p must be invariant under the compact n-torus action generated by the closure of the flow of the imaginary part of Z. This puts severe restrictions on the possibilities for such solitons.

At the conclusion of Section § 3, I discuss the local generality problem near a singular point of Z and explain how it can best be viewed as an elliptic boundary value problem of a certain type, but do not go into any further detail. A fuller discussion of this case may perhaps be undertaken at a later date.

1.3. The positive case. — In Section § 4, I turn to an interesting special case: The case where g is complete, the Ricci curvature is positive, and the scalar curvature $R(g)$ attains its maximum at some (necessarily unique) point $p \in M$.

This case has been studied before by Cao and Hamilton [**6**], who proved that this point p is a minimum of the Ricci potential f, that f is a proper plurisubharmonic exhaustion function on M (which is therefore Stein), and that, moreover, the Killing field $J(\nabla f)$ has a periodic orbit on 'many' of its level sets.

(1) Of course, it is by no means true that every holomorphic vector field is holomorphically linearizable at each of its singular points.