Astérisque

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Astérisque, tome 321 (2008), p. 5-29

<http://www.numdam.org/item?id=AST_2008__321__5_0>

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EINSTEIN METRICS AND MAGNETIC MONOPOLES

by

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For Jean Pierre Bourguignon on his 60th birthday

Abstract. — We investigate the geometry of the moduli space of centred magnetic monopoles on hyperbolic three-space, and derive using twistor methods some (incomplete) quaternionic Kähler metrics of positive scalar curvature. For the group SU(2) these have an orbifold compactification but we show that this is not the case for SU(3).

Résumé **(Métriques d'Einstein et monopoles magnétiques).** — Nous étudions la géométrie des espaces de modules des monopoles maghétiques sur le 3-espace hyperbolique et nous en dérivons quelques métriques kählériennes quaternioniques (incomplètes) de courbure scalaire positive, en utilisant des méthodes *twistor*. Celles-ci ont une compactification orbifolde pour le groupe SU(2) et nous montrons qu'il n'en est rien dans le cas du groupe SU(3).

1. Introduction

Over 20 years ago Jean Pierre Bourguignon and I were part of the team helping Arthur Besse to produce a state-of-the-art book on Einstein manifolds [**3**]. As might have been expected, the subject proved to be a moving target, and the contributors had to quickly assemble a number of appendices to cover material that came to light after all the initial planning. The last sentence of the final appendix refers to: "hyperkählerian metrics on finite dimensional moduli spaces", and so it seems appropriate to write here about some of the results which have followed on from that, and some questions that remain to be answered.

There is by now a range of gauge-theoretical moduli spaces which have natural hyperkähler metrics: the moduli space of instantons on $\mathbf{R}^4$ or the 4-torus or a K3

2000 Mathematics Subject Classification. — 53C28, 53C26, 51N30.
Key words and phrases. — Hyperkähler, quaternionic Kähler, moduli space, monopole, Bogomolny equations, spectral curve.

surface [**16**], magnetic monopoles on $\mathbf{R}^3$ [**2**] and Higgs bundles on a Riemann surface [**12**]. The latter structure features prominently in the recent work of Kapustin and Witten on the Geometric Langlands correspondence [**15**]. Some of these metrics, in low dimensions, can be explicitly calculated, but even when this is not possible, the fact that these spaces are moduli spaces enables us to observe some geometrical properties which reflect their physical origin. In this paper we shall concentrate on the case of magnetic monopoles.

For monopoles in Euclidean space $\mathbf{R}^3$, there exist in certain cases explicit formulae (for example [**5**]), but in general we cannot write the metric down. Instead we can seek a geometrical means to describe the metrics; such a technique is provided by the use of twistor spaces, spectral curves and the symplectic geometry of the space of rational maps. This is documented in [**2**]. We review this in Section 2, drawing on new approaches to the symplectic structure.

We then shift attention to the hyperbolic version. The serious study of monopoles in hyperbolic space $\mathbf{H}^3$ was initiated long ago by Atiyah [**1**], who showed that there were many similarities with the Euclidean case. Yet the differential-geometric structure of the moduli space is still elusive, despite recent efforts [**18**], [**19**]. One would expect some type of quaternionic geometry which in the limit where the curvature of the hyperbolic space becomes zero approaches hyperkähler geometry. In Section 3 we give one approach to this, and show, following [**17**], how to resolve one of the problems that arises in attempting this – assigning a centre to a hyperbolic monopole.

The other problem, concerning a real structure on the putative twistor space, can currently be avoided only in the case of charge 2 and in Section 4 we produce, for the groups SU(2) and SU(3), quaternionic Kähler metrics on the moduli spaces of centred hyperbolic monopoles, generalizing the Euclidean cases computed in [**2**] and [**8**]. These metrics are expressed initially in twistor formalism, using the holomorphic contact geometry of certain spaces of rational maps, but we obtain some very explicit formulae as well.

For SU(2), these concrete self-dual Einstein metrics, originally introduced in [**14**], have nowadays found a new life in the area of 3-Sasakian geometry, Kähler-Einstein orbifolds and manifolds of positive sectional curvature. We consider briefly these aspects in the final section, and suggest where new examples might be found.

2. Euclidean monopoles

All of the hyperkähler moduli spaces mentioned above arise through the hyperkähler quotient construction. Recall that a hyperkähler metric on a manifold M^{4n} is defined by three closed 2-forms $\omega_1, \omega_2, \omega_3$ whose joint stabilizer at each point is conjugate to $Sp(n) \subset GL(4n, \mathbf{R})$. If a Lie group G acts on M, preserving the forms, then there usually exists a hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$. The quotient construction is the statement that the induced metric on $\mu^{-1}(0)/G$ is also hyperkähler.

For the moduli space of monopoles we use an infinite-dimensional version of this. The objects consist of connections A on a principal G-bundle over $\mathbf{R}^3$ together with a Higgs field ϕ, a section of the adjoint bundle. There are boundary conditions at infinity [**2**], in particular that $\|\phi\| \sim 1 - k/2r$, which imply that the connection on the sphere of radius R approaches a standard homogeneous connection as $R \to \infty$. The manifold M to which we apply the quotient construction then consists of pairs (A, ϕ) which differ from this standard connection by terms which decay appropriately, and in particular are in $\mathcal{L}^2$. This is formally an affine flat hyperkähler manifold where the closed forms ω_i are given by

$$\omega_i((\dot{A}_1, \dot{\phi}_1), (\dot{A}_2, \dot{\phi}_2)) = \int_{\mathbf{R}^3} dx_i \wedge \operatorname{tr}(\dot{A}_1 \dot{A}_2) + \int_{\mathbf{R}^3} *dx_i \wedge [\operatorname{tr}(\dot{\phi}_1 \dot{A}_2) - \operatorname{tr}(\dot{\phi}_2 \dot{A}_1)].$$

For the symplectic action of a group we take the group of gauge transformations which approach the identity at infinity suitably fast.

The zero set of the moment map in this case consists of solutions to the Bogomolny equations $F_A = *d_A\phi$, and the hyperkähler quotient is a bundle over the true moduli space of solutions – it is a principal bundle with group the automorphisms of the homogeneous connection at infinity. This formal framework has to be supported by analytical results of Taubes to make it work rigorously.

When $G = \mathrm{SU}(2)$, the connection on a large sphere has structure group $U(1)$ and Chern class k, which is called the monopole charge. The hyperkähler quotient is a manifold of dimension $4k$ which is a circle bundle over the true moduli space. It has a complete metric which is invariant under the Euclidean group and the circle action (completeness comes from the Uhlenbeck weak compactness theorem, one use of gauge theoretical results to shed light on metric properties). The gauge circle action in fact preserves the hyperkähler forms $\omega_1, \omega_2, \omega_3$, and its moment map defines a centre in $\mathbf{R}^3$. The $(4k - 4)$-dimensional hyperkähler quotient can then be thought of as the moduli space of centred monopoles.

For charge 2, by using a variety of techniques [**2**], one can determine the metric explicitly. It has an action of SO(3) and may be written as

$$g = (abc)^2 d\eta^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{1}$$

where

$$ab = -2k(k')^2 K \frac{dK}{dk} \qquad bc = ab - 2(k'K)^2 \qquad ca = ab - 2(k'K)^2$$

$$\eta = -K'/\pi K \qquad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

and $\sigma_1, \sigma_2, \sigma_3$ are the standard left-invariant forms on SO(3).

Differentiably, this manifold can be understood in terms of the unit sphere in the irreducible 5-dimensional representation space of SO(3). For each axis there is, up to a scalar multiple, a unique axially symmetric vector in this representation and these trace out two copies of $\mathbf{R}\mathrm{P}^2 \subset S^4$. The centred moduli space is the complement of one of these, the removed point being the axis joining two widely separated monopoles. The other $\mathbf{R}\mathrm{P}^2$ parametrizes axially symmetric monopoles, which are (for any value of charge) uniquely determined by their axis.

For the group $G = \mathrm{SU}(3)$ we consider a Higgs field which asymptotically has two equal eigenvalues. On a large sphere the eigenspace is a rank two bundle with first Chern class k, again called the charge. When $k = 2$, Dancer computed this metric [**7**]. For centred monopoles it is eight-dimensional with an $\mathrm{SO}(3)\times\mathrm{PSU}(2)$ action, the first factor from the geometric action of rotations and the second from the automorphisms of the connection at infinity. Explicitly it can be written as follows:

$$g = \frac{1}{4}\sum_i (x(1+px)m_i m_i + y(1+py)n_i n_i + 2pxym_i n_i)$$

where

$$\begin{aligned}
m_1 &= -f_1 df_1 + f_2 df_2 \quad m_2 = (f_1^2 - f_2^2)\sigma_3 \\
m_3 &= \frac{1}{px}[(pyf_3^2 - (1+py)f_1^2)\sigma_2 + f_3 f_1 \Sigma_2] \\
m_4 &= -\frac{1}{1+px+py}[(pyf_3^2 - (1+py)f_2^2)\sigma_1 + f_2 f_3 \Sigma_1] \\
n_1 &= \frac{1}{py}(-pxf_2 df_2 + (1+px)f_1 df_1) \\
n_2 &= \frac{1}{py}[((1+px)f_1^2 - pxf_2^2)\sigma_3 - f_1 f_2 \Sigma_3] \\
n_3 &= (f_1^2 - f_3^2)\sigma_2 \\
n_4 &= \frac{1}{1+px+py}[(pxf_2^2 - (1+px)f_3^2)\sigma_1 + f_2 f_3 \Sigma_1]
\end{aligned}$$

with σ_i, Σ_i invariant one-forms on $\mathrm{SO}(3) \times \mathrm{SU}(2)$, and

$$f_1 = -\frac{D\,\mathrm{cn}(3D,k)}{\mathrm{sn}(3D,k)} \quad f_2 = -\frac{D\,\mathrm{dn}(3D,k)}{\mathrm{sn}(3D,k)} \quad f_3 = -\frac{D}{\mathrm{sn}(3D,k)},$$

$$x = \frac{1}{D^3}\int_0^{3D} \frac{\mathrm{sn}^2(u)}{\mathrm{dn}^2(u)}du \quad y = \frac{1}{D^3}\int_0^{3D} \mathrm{sn}^2(u)du.$$

and $p = f_1 f_2 f_3$ for $D < 2K/3$.

Clearly there are limits to extracting information from formulae like these. Nevertheless, the restriction to certain submanifolds can be useful.