Astérisque

NGAIMING MOK

Geometric structures on uniruled projective manifolds defined by their varieties of minimal rational tangents

Astérisque, tome 322 (2008), p. 151-205
<http://www.numdam.org/item?id=AST_2008__322__151_0>

Astérisque
322, 2008, p. 151–205

GEOMETRIC STRUCTURES ON UNIRULED PROJECTIVE MANIFOLDS DEFINED BY THEIR VARIETIES OF MINIMAL RATIONAL TANGENTS

by

Ngaiming Mok

Abstract. — In a joint research programme with Jun-Muk Hwang we have been investigating geometric structures on uniruled projective manifolds, especially Fano manifolds of Picard number 1, defined by varieties of minimal rational tangents associated to moduli spaces of minimal rational curves. In this article we outline a heuristic picture of the geometry of Fano manifolds of Picard number 1 with non-linear varieties of minimal rational tangents, taking as hints from prototypical examples such as those from holomorphic conformal structures. On an open set in the complex topology the local geometric structure associated to varieties of minimal rational tangents is equivalently given by families of local holomorphic curves marked at a variable base point satisfying certain compatibility conditions. Differential-geometric notions such as (null) geodesics, curvature and parallel transport are a source of inspiration in our study. Formulation of problems suggested by this heuristic analogy and their solutions, sometimes in a very general context and at other times applicable only to special classes of Fano manifolds, have led to resolutions of a series of well-known problems in Algebraic Geometry.

Résumé **(Structures géométriques sur des variétés projectives uniréglées définies par leurs variétés de tangentes rationnelles minimales)**
Dans un programme de recherche avec Jun-Muk Hwang nous avons étudié des structures géométriques sur les variétés projectives uniréglées, en particulier les variétés de Fano de nombres de Picard égaux à 1, definies par les variétés de tangentes rationnelles minimales associées aux espaces de modules de courbes rationnelles minimales. Dans cet article nous esquissons un dessin heuristique sur la géométrie des variétés de Fano de nombres de Picard égaux à 1 dont les variétés de tangentes rationnelles minimales sont non linéaires, en prenant comme prototypes les exemples tels ques les structures conformes holomorphes. Dans un ouvert par rapport à la topologie complexe, la structure géométrique associée aux variétés de tangentes rationnelles minimales équivaut aux données de familles de courbes holomorphes locales marquées à un point de base variable vérifiant des conditions de compatibilité. Des notions de la géometrie différentielle comme les géodésiques (nulles), la courbure et le transport parallèle constituent une source d'inspiration dans notre étude. Des formulations de problèmes suggérés par cette analogie heuristique et leurs solutions, parfois dans

2000 Mathematics Subject Classification. — 14J45, 32M15, 32H02, 53C10.
Key words and phrases. — Geometric structure, minimal rational curve, variety of minimal rational tangents, tangent map, analytic continuation, Cauchy characteristic, curvature, prolongation, parallel transport, nef tangent bundle, distribution, differential system, deformation rigidity.

Research partially supported by a CERG grant of the Research Grants Council of Hong Kong.

un contexte très générale et parfois applicables seulement aux classes de variétés de Fano spéciales, ont conduit a des résolutions d'une série de problèmes bien connus en géométrie algébrique.

1. Introduction

1.1. Background and motivation. — In 1979, Mori [**45**] established the fundamental existence result on rational curves on a projective manifold where the canonical line bundle is not numerically effective, thereby resolving the Hartshorne Conjecture. When the manifold is Fano, Miyaoka-Mori [**38**] (1986) proved that the manifold is uniruled. In a joint research programme undertaken with Jun-Muk Hwang, we have been studying algebro-geometric and complex-analytic problems on uniruled projective manifolds basing on geometric objects arising from special classes of rational curves, viz., minimal rational curves. In this article the author would like to highlight some geometric aspects of the underlying theory.

Given a uniruled projective manifold X and fixing an ample line bundle L, by a minimal rational curve we will mean a free rational curve of minimal degree with respect to L among all free rational curves. A connected component $\mathcal{K}$ of the space of minimal rational curves will be called a minimal rational component. In practice we will fix a minimal rational component $\mathcal{K}$ and consider only minimal rational curves belonging to $\mathcal{K}$. Associated to $\mathcal{K}$, there is the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, where $\rho : \mathcal{U} \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, and $\mu : \mathcal{U} \to X$ is the evaluation map. In connection with $\mathcal{U}$ there is the tangent map $\tau : \mathcal{U} \to \mathbb{P}T_X$. For a minimal rational curve C marked at $x \in X$ and immersed at the marking, and for α denoting a nonzero vector tangent to C at the marking, the tangent map associates to the marked point the element $[\alpha] \in \mathbb{P}T_x(X)$. For a general point $x \in X$ we define the variety of minimal rational tangents (VMRT) $\mathcal{C}_x$ at x to be the strict transform of the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$. The basic set-up of our study takes place on the total space of the double fibration given by the universal family $\rho : \mathcal{U} \to X$, $\mu : \mathcal{U} \to X$, equipped with the tangent map $\tau : \mathcal{U} \to \mathbb{P}T_x(X)$ and the fibered space $\pi : \mathcal{C} \to X$ of VMRTs. The overriding question is the extent to which a uniruled projective manifold X is determined by its VMRTs.

Given a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component $\mathcal{K}$, and a connected open subset $U \subset X$ in the complex topology, we consider $(U, \mathcal{C}|_U)$ as a complex manifold equipped with a geometric structure. Here the term 'geometric structure' is understood by analogy to standard examples. As a prototype in the context of smooth manifolds, a real m-dimensional Riemannian manifold (M, g) can be understood as one equipped with a reduction of the frame bundle from the structure group $\mathrm{GL}(m,\mathbb{R})$ to $\mathrm{O}(m)$. In the context of complex manifolds, a simplest example of a *holomorphic* geometric structure relevant to the study of uniruled projective manifolds is the case of holomorphic conformal structures, alias hyperquadric structures. A holomorphic conformal structure on an n-dimensional complex manifold X determines at every point $x \in X$ its null-cone, defining equivalently a holomorphic fiber subbundle $\mathcal{Q} \subset \mathbb{P}T_X$ consisting of fibers $\mathcal{Q}_x$ isomorphic to an $(n-2)$-dimensional hyperquadric. It corresponds to a reduction of the holomorphic frame bundle from $\mathrm{GL}(n;\mathbb{C})$ to $\mathbb{C}^* \cdot \mathrm{O}(n;\mathbb{C})$, and this reduction is completely determined by $\mathcal{Q} \subset \mathbb{P}T_X$. When $X = Q^n$, the n-dimensional hyperquadric, $\mathcal{Q}_x$ agrees with the VMRT $\mathcal{C}_x$, and by analogy we speak of the geometric structure on a uniruled projective manifold $(X,\mathcal{K})$ equipped with a minimal rational component as defined by its fibered space $\pi : \mathcal{C} \to X$ of VMRTs. As our geometric study of VMRTs are in many cases motivated by differential-geometric consideration, especially in relation to global properties that can be captured by local differential-geometric information, we will be considering a general point $x \in X$, and the local geometric structure defined by the germ of the fibered space $\pi : \mathcal{C} \to X$ at x, equivalently the restriction $\pi|_U : \mathcal{C}|_U :\to U$ to arbitrarily small Euclidean open neighborhoods U of x.

1.2. A heuristic picture. — While a substantial part of our programme applies generally to any uniruled projective manifold, our focus of investigation has been primarily on those of Picard number 1. These manifolds, which are necessarily Fano, are not amenable to further reduction by means of extremal rays in Mori theory, and as such they are called 'hard nuts' among Fano manifolds in Miyaoka [**36**]. Our geometric theory on uniruled projective manifolds based on VMRTs serve in particular as a basis for a systematic study of all Fano manifolds of Picard number 1. There emerges a dichotomy between those for which the VMRT at a general point is the union of finitely many projective linear subspaces and the rest. We will say that $(X,\mathcal{K})$ has linear VMRTs in the former case and non-linear VMRTs otherwise. The linear case includes those for which the VMRT at a general point is 0-dimensional, where the fibered space $\pi : \mathcal{C} \to X$ gives rise to a geometry on X resembling that of web geometry. We will discuss in this article exclusively the non-linear case and refer the reader to Hwang-Mok [**20**] (2003) for results in the case of 0-dimensional VMRTs, and to Hwang [**13**] (2007) for a problem which necessitates the study of the hypothetical case of linear VMRTs of higher dimensions.

At this stage of the investigation we have the following heuristic picture in the case of non-linear VMRTs. The universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ associated to the minimal rational component $\mathcal{K}$ gives rise via the tangent map to a tautological multifoliation on the fibered space $\pi : \mathcal{C} \to X$ of VMRTs, and the 'local' geometric structure $(U, \mathcal{C}|_U)$ on open subsets $U \subset X$ in the complex topology corresponds to the data of families of local holomorphic curves marked at points $x \in U$. The local holomorphic curves are then solutions to a system of partial differential equations which in the case of holomorphic conformal structures correspond to the null geodesics. We may think of the local holomorphic curves as analogues of (null) geodesics. The fact that these 'geodesics' can be extended to minimal rational curves on $(X, \mathcal{K})$ should impose serious constraints on the underlying geometric structure. In the case of the holomorphic conformal structure on the hyperquadric, the splitting type of the tangent bundles on minimal rational curves is enough to force the vanishing of the holomorphic Bochner-Weyl tensor and thus to force flatness of the structure. In the general case of $(X, \mathcal{K})$, for a general $\mathcal{K}$-minimal rational curve the normal bundle has only direct summands of degree 1 or 0. Such a rational curve, to be called a standard rational curve, resembles minimal rational curves on a hyperquadric, and there ought to be partial 'flatness' of the geometric structure of $(X, \mathcal{C})$ along standard rational curves which places serious restrictions on geometric structures that can possibly arise from VMRTs. The heuristic analogy between minimal rational curves and (null) geodesics also goes further as the former should serve to propagate geometric information from a germ of geometric structure to the ambient Fano manifold X of Picard number 1. In this case, any two general points can be connected by a chain of minimal rational curves, and the bad set of 'inaccessible points' must be of codimension ≥ 2.

A further geometric concept that ought to play an important role in the study of geometric structures defined by VMRTs is the notion of parallel transport along a standard rational curve. In the special case of irreducible Hermitian symmetric spaces of the compact type the VMRTs are invariant under parallel transport with respect to any choice of a canonical Kähler-Einstein metric. For Fano manifolds of Picard number 1, endowed with geometric structures arising from VMRTs but without privileged local holomorphic connections, the only general source for the notion of parallel transport arises from splitting types over minimal rational curves. In this direction it is found that for the germ of families of VMRTs along the tautological lifting $\widehat{C}$ of a standard rational curve, the second fundamental in the fiber directions can be identified as a section of a flat bundle over C, and as such one can speak of the parallel transport of second fundamental forms along a standard rational curve.

Other than geometric structures defined by VMRTs, in important classes of Fano manifolds X of Picard number 1 there are additional underlying structures with differential-geometric meaning. These are the cases where the VMRTs are positive-dimensional, irreducible and linearly degenerate at a general point. They span distributions which give rise to differential systems by taking Lie brackets. The study of this class of manifolds, which is particularly important for questions on deformation rigidity, reveals an intimate link between issues of integrability and projective-geometric properties of the VMRT at a general point.